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BOX–LINE RELAXATION SCHEMES FOR SOLVING THE STEADY INCOMPRESSIBLE NAVIER–STOKES EQUATIONS USING SECOND–ORDER UPWIND DIFFERENCING*1)

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Abstract

We extend the SCGS smoothing procedure (Symmetrical Collective Gauss–Seidel relaxation) proposed by S. P. Vanka[4], for multigrid solvers of the steady viscous incompressible Navier–Stokes equations, to corresponding line–wise versions. The resulting relaxation schemes are integrated into the multigrid solver based on second–order upwind differencing presented in [5]. Numerical comparisons on the efficiency of point–wise and line–wise relaxations are presented.

1. Introduction

The convection–diffusion behaviour of the viscous incompressible Navier–Stokes equations is a main source of difficulties in the numerical solution. When discretizing the equations using finite difference schemes, upwind or hybrid schemes are usually used on the convection terms for ensuring the stability of the discrete system [1]. The first–order upwind differencing has proved to be inadequate for the incompressible Navier–Stokes equations with large Reynolds numbers, although the resulting discrete systems are very stable and easily solved. In [5], we constructed a multigrid solver based on second–order upwind differencing and we adapted the SCGS relaxation, which was originally proposed for hybrid schemes, as the smoothing procedure. It gives good discrete solutions and the convergence rate is comparable to (even faster than) the same multigrid solver using first–order upwind differencing when the cell Reynolds number is not very large. There are two main disadvantages for the SCGS relaxation: 1) with second–order upwind differencing, it is difficult to obtain convergence for very large Reynolds numbers ($R \geq 2000$) and the convergence rate is sensitive to the relaxation factor; 2) it fails for strongly anisotropic problems, e.g., when the aspect ratio of the grid cells is not close to 1, so it is not suitable on non–uniform grids.

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In this paper, we give two line–wise extensions to the SCGS relaxation for the second–order upwind scheme and we make some numerical comparisons on the convergence rate of different relaxation methods.

2. Discretization

The dimensionless steady viscous incompressible Navier–Stokes equations in a 2D domain Ω can be formulated as follows:

$$\begin{cases} -\dfrac{1}{R}\Delta u + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + \dfrac{\partial p}{\partial x} = f_1, \\ -\dfrac{1}{R}\Delta v + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + \dfrac{\partial p}{\partial y} = f_2, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \\ + \text{Boundary conditions} \end{cases} \qquad (1)$$

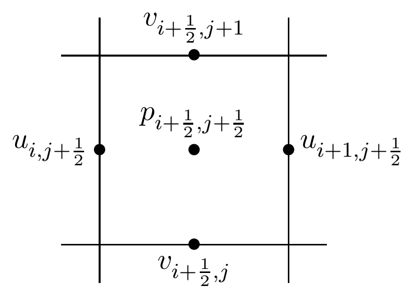

Fig. 1. Location of unknowns

where (u, v) is the velocity, p the pressure, R the Reynolds number and (f_1, f_2) denotes the external force.

We discretize Equation (1) on uniform staggered grids (MAC grid). The location of different variables and the corresponding discrete equations on the cell (i, j) is shown by Fig. 1 (in which the index (i, j) corresponds to the grid point $(i\Delta x, j\Delta y)$).

The convection terms in (1) are discretized using second–order upwind differencing. For example, the term $v\dfrac{\partial u}{\partial y}$ on the point $(i\Delta x, (j + \frac{1}{2})\Delta y)$ is discretized by:

$$\begin{cases} \dfrac{v_{i,j+\frac{1}{2}}}{2\Delta y}(3u_{i,j+\frac{1}{2}} - 4u_{i,j-\frac{1}{2}} + u_{i,j-\frac{3}{2}}), & \text{if } v_{i,j+\frac{1}{2}} \geq 0, \\ -\dfrac{v_{i,j+\frac{1}{2}}}{2\Delta y}(3u_{i,j+\frac{1}{2}} - 4u_{i,j+\frac{3}{2}} + u_{i,j+\frac{5}{2}}), & \text{if } v_{i,j+\frac{1}{2}} < 0 \end{cases}$$

where the term $v_{i,j+\frac{1}{2}}$, which is not defined on the grid points, is computed by bilinear interpolation:

$$v_{i,j+\frac{1}{2}} = \frac{1}{4}(v_{i-\frac{1}{2},j} + v_{i-\frac{1}{2},j+1} + v_{i+\frac{1}{2},j} + v_{i+\frac{1}{2},j+1}).$$

All other terms are discretized by standard central differencing. For details on the discretization and treatment near the boundaries we refer to [5].

3. Relaxation Schemes