Journal of Computational Mathematics, Vol.10, No.2, 1992, 181–193.

RATIONAL INTERPOLATION FOR STIELTJES FUNCTIONS*[1)]

Xu Guo-liang
(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

The continuity conclusions about rational Hermite interpolating functions are given under some conditions. On that basis, we establish the convergence results for the paradiagonal sequences of the rational interpolants for Stieltjes functions and Hamburger functions.

§1. Introduction

Let $\{x_i\}_{i=0}^{\infty} \subset [a,b]$ and $f \in C[a,b]$ be properly smooth. Given integers m, n, we consider the following problem: Find $R \in R_{mn} = \{u = p/q: \ p \in H_m, \ q \in H_n\}$ such that

$$R^{\sigma_i}(x_i) = f^{\sigma_i}(x_i), \qquad i = 0,1,\cdots,m+n, \tag{1.1}$$

where H_l denotes the class of all polynomials of degree at most l and $\sigma_i + 1$ is the multiplicity of x_i in $\{x_0, x_1, \cdots, x_i\}$. Relating to the above problem, we introduce a linearized problem as follows. Find $(P,Q) \in H_m \times H_n$ such that

$$(Qf - P)^{(\sigma_i)}(x_i) = 0, \qquad i = 0,1,\cdots,m+n. \tag{1.2}$$

Complete results about the solvability of the two problems can be found in [5]. For our purpose in this paper, we introduce the following conclusions.

Theorem 1.1 ([5],[6]). (i) *Problem* (1.1) *is solvable iff*

$$\begin{aligned} &\operatorname{rank} C(m-1, n-1, x_0, x_1, \cdots, x_{m+n}) \\ &\quad = \operatorname{rank} C(m-1, n-1, x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+m}), \\ &\quad i = 0,1,\cdots,m+n. \end{aligned} \tag{1.3}$$

* Received January 12, 1989.
1) The Project Supported by National Science Foundation of China for Youth.

(ii) *Let* (P^*, Q^*) *be a solution of problem* (1.2) *with minimum degree. Then* $\partial P^* = m$ *if and only if the matrix* $C(m-1, n, x_0, \cdots, x_{m+n})$ *is nonsingular;* $\partial Q^* = n$ *if and only if* $C(m, n-1, x_0, \cdots, x_{m+n})$ *is nonsingular.*

(iii) *The interpolation operator* T_{mn}, *for which* $T_{mn}(x_0, \cdots, x_{m+n}, f) = (P^*, Q^*)$, *is continuous at* $(x_0, \cdots, x_{m+n}, f)$ *if and only if* (P^*, Q^*) *is non-degenerate.*

The matrices $C(p, q, t_0, \cdots, t_k)$ used in the above theorem are defined as

$$C(p,q,t_0,\cdots,t_k) = \begin{bmatrix} v^{\sigma_0}(p,q,t_0) \\ v^{\sigma_1}(p,q,t_1) \\ \vdots \\ v^{\sigma_k}(p,q,t_k) \end{bmatrix},$$

where

$$v(p,q,t) = [1, t, \cdots, t^p, f(t), tf(t), \cdots, t^q f(t)]$$

and $\sigma_i + 1$ is the multiplicity of t_i in $\{t_0, t_1, \cdots, t_i\}$.

Let

$$H(m,i,j,t_0,\cdots,t_{m+j}) = \begin{bmatrix} f_{0,m} & f_{1,m} & \cdots & f_{i,m} \\ f_{0,m+1} & f_{1,m+1} & \cdots & f_{i,m+1} \\ \cdots & \cdots & \cdots & \cdots \\ f_{0,m+j} & f_{1,m+j} & \cdots & f_{i,m+j} \end{bmatrix},$$

where f_{ij} is the divided difference of f at $t_i, t_{i+1}, \cdots, t_j$. Then we have

$$\operatorname{rank} C(p,q,t_0,\cdots,t_k) = p + 1 + \operatorname{rank}\ H(p+1, q, k-p-1, t_0, \cdots, t_k). \tag{1.4}$$

Hence the conclusions (i) and (ii) in Theorem 1.1 can be restated as follows:

(i) Problem (1.1) is solvable if and only if

$$\begin{aligned} &\operatorname{rank} H(m, n-1, n, x_0, \cdots, x_{m+n}) \\ &\quad = \operatorname{rank} H(m, n-1, n-1, x_0, \cdots, x_{i-1}, x_{i+1}, \cdots x_{m+n}), \quad i = 0, 1, \cdots, m+n. \end{aligned}$$

(ii) $\partial(P^*) = m \iff H(m, n, n, x_0, \cdots, x_{m+n})$ is nonsingular; $\partial(Q^*) = n \iff H(m+1, n-1, n-1, x_0, \cdots, x_{m+n})$ is nonsingular.

For the Cauchy interpolation problem, i.e., the interpolation points x_i are mutually distinct, the continuity results of the interpolation function $u_{mn} := R$ to f and the conclusions about the position of poles of u_{mn} are obtained by Braess in [2],[3]. In this paper, we first generalize these results to the Hermite case by a similar approach, and then establish convergence results for paradiagonal sequences of the rational interpolations for Stieltjes functions and Hamburger functions. For Padé approximants, similar convergence results can be found in [1].