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ON THE STABILITY OF BIQUADRATIC-BILINEAR VELOCITY-PRESSURE FINITE ELEMENTS *1)

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Abstract

In this paper, the stability of biquadratic-bilinear velocity-pressure finite elements are discussed for the stationary Stokes problem. It is proved that there exist constants $c, c' > 0$ independent of h such that

$$c'h \geq \inf_{\substack{q_h \in Q_h \\ q_h \neq 0}} \sup_{\substack{\vec{v}_h \in V_h \\ \vec{v}_h \neq 0}} \frac{(\operatorname{div} \vec{v}_h, q_h)}{\|q_h\|_h |\vec{v}_h|_1} \geq ch$$

Hence a question in [1] is answered.

§1. Introduction

Let us consider the mixed finite element methods for the stationary Stokes problem.It is well known[1],[2] that many usual finite elements do not satisfy the discrete classical Babuska-Brezzi condition and there exist the checker-board modes. G.F. Carey and R. Krishnan[1] proposed the analogy condition,i.e.,there exists β_h dependent on h,such that

$$\sup_{\substack{\vec{v}_h \in V_h \\ \vec{v}_h \neq 0}} \frac{(\operatorname{div} \vec{v}_h, q_h)}{|\vec{v}_h|_1} \geq \beta_h \|q_h\|_h, \quad \forall q_h \in Q_h \tag{1.1}$$

where $V_h \in [H_0^1(\Omega)]^2$ and $Q_h \in L_0^2(\Omega)$ are two finite element spaces, $H_0^1(\Omega)$ denotes the usual Sobolev space with the seminorm $|\cdot|_1$, $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$, and

$$\|q_h\|_h = \inf_{q_h^* \in Ker B_h^*} \|q_h + q_h^*\|_0 \tag{1.2}$$

$$Ker B_h^* = \{q_h \in Q_h : (q_h, div \vec{v}_h) = 0, \forall \vec{v}_h \in V_h\} \tag{1.3}$$

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Obviously, it is important to estimate the order of β_h in condition (1.1). G. F. Carey et al[1] have got $\beta_h = O(h)$ and asked whether this estimate is optimal. In this paper, we discuss the biquadratic-bilinear velocity-pressure (Q_2/Q_1) finite elements and prove that the result $\beta_h = O(h)$ in (1.1) is optimal. In addition, it is shown that the result $\beta_h = O(h^{\frac{1}{2}})$ in [4] is not true.

§2. Q_2/Q_1 Finite Elements

Assume the bounded region Ω is just the unit square $0 < x, y < 1$. Let J_h be a subdivision of Ω, i.e., Ω is divided into subsquares of side $h = \frac{1}{N}$ ($N > 1$ integer). Denote subsquares

$$K_{ij} = [ih, (i+1)h] \times [jk, (j+1)h]$$

for $i, j = 0, 1, 2, \cdots, N-1$.

Define the Q_2/Q_1 finite element spaces:

$$V_h = \{\vec{v}_h \in (C^0(\bar{\Omega}))^2; \vec{v}_h|_{K_{ij}} \in [Q_2(K_{ij})]^2\} \cap [H_0^1(\Omega)]^2, \tag{2.1}$$

$$Q_h = \{q_h \in L_0^2(\Omega); q_h|_{K_{ij}} \in Q_1(K_{ij})\}, \ i, j = 0, 1, 2, \cdots, N-1 \tag{2.2}$$

where $Q_2(K)$ and $Q_1(K)$ denote the biquadratic polynomial space and bilinear polynomial space on K and $L_0^2(\Omega) = \left\{q \in L_2(\Omega), \int_\Omega q dx = 0\right\}$.

For two finite element spaces V_h, and Q_h defined by (2.1) and (2.2) we shall show the following result.

Theorem 1. *There exist constants $c, c' > 0$ independent of h such that*

$$c'h \geq \inf_{q_h \in Q_h} \sup_{\vec{v}_h \in V_h} \frac{(\operatorname{div} \vec{v}_h, q_h)}{|\vec{v}_h|_1 \|q_h\|_h} \geq ch.$$

The inequality on the right has been proved in [1]. Here we will deal with only the inequality on the left. Let P_{ij}^e denote the standard biquadratic basis function,equal to 1 at A_{ij}^e and zero at the other nodes,where $A_{ij}^e (e = 1, 2, \cdots, 9)$ denote the nodes in K_{ij} (see Fig.1)