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# HIGH-ORDER COMPACT DIFFERENCE METHODS FOR SIMULATING WAVE PROPAGATIONS IN EXCITABLE MEDIA

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**Abstract.** In this paper, we present a study of some high-order compact difference schemes for solving the Fitzhugh-Nagumo equations governed by two coupled time-dependent nonlinear reaction diffusion equations in two variables. Solving the Fitzhugh-Nagumo equations is quite challenging, since the equations involve spatial and temporal dynamics in two different scales and the solutions exhibit shock-like waves. The numerical schemes employed have sixth order accuracy in space, and fourth order in time if the fourth order Runge-Kutta method is adopted for time marching. To improve efficiency, we also propose an ADI scheme (for two dimensional problems), which has second order accuracy in time. Numerical results are presented for plane wave propagation in one dimension and spiral waves for two dimensions.

**Key words.** Spiral waves, excitable medium, FitzHugh-Nagumo equations, compact difference methods.

## 1. Introduction

In this work, we consider the following two-dimensional (2D) Fitzhugh–Nagumo type model (cf. [1, 2]) for the description of waves in excitable media given by

$$\frac{\partial u}{\partial t} = \nabla^2 u + f(u,v), \quad \frac{\partial v}{\partial t} = g(u,v), \tag{1}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denotes the Laplace operator, $u = u(x,y,t)$ and $v = v(x,y,t)$ are the so-called excitation variable and recovery variable, respectively. The functions $f(u,v)$ and $g(u,v)$ represent the local reaction kinetics of the species. Here we adopt the simplified Barkley model given by [3]

$$f = \frac{1}{\epsilon} u(1-u)(u - \frac{v+b}{a}), \quad g = u - v, \tag{2}$$

where the constants $a$ and $b$ control the excitability threshold and duration, and $\epsilon$ determines the time scale of the fast variable $u$. Usually, $\epsilon$ is selected quite small such that the time scale of $u$ is several orders of magnitude faster than that of $v$. A larger value of $a$ would increase the excitation variable duration, whereas a larger ratio $b/a$ increases the excitation threshold.

To make the problem (1) complete, we assume that (1) hold true for $(x,y,t) \in \Omega \times [0,T]$, where $\Omega \subset R^2$ is an open, bounded, connected polygonal domain with boundary $\partial\Omega$, along with zero-flux boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{3}$$

where $n$ is the unit outer normal vector of $\partial\Omega$, and appropriate initial conditions

$$u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y) \quad \text{in } \Omega. \tag{4}$$

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The FitzhughNagumo model (1)–(4) is the most widely used mathematical model of excitation and propagation of impulse in excitable media such as nerve membranes. Over the years, there have been many studies devoted to this model and its many variations (e.g., [4] and references therein). Due to the complexity of the problem, numerical simulation plays a very important role in studing the FitzhughNagumo model. For example, Barkley [3] presented a simple and efficient finite difference algorithm (attached with a complete subroutine) for solving the 2D FitzhughNagumo equations. Later, Olmos and Shizgal [5] proposed a Chebyshev multidomain method, and a new finite difference method for solving both 1D and 2D FitzhughNagumo equations. Ramos [6] numerically studied the propagation of spiral waves in 2D reactivediffusive media. In [7], Amdjadi proposed a numerical method for testing the dynamics and stability of spiral waves in excitable media. Bürger, Ruiz-Baier and Schneider [8] presented some fully space-time adaptive multiresolution methods based on the finite volume method and Barkley's method for simulating the complex dynamics of waves in excitable media. In [9], Dehghan and Fakhar-Izadi developed two pseudospectral methods based on Fourier series and rational Chebyshev functions to solve the 1D Nagumo equation.

The main objective of the present paper is to introduce the high-order compact difference method ([10, Ch.5] and references) to simulate the wave propagation problem in excitable media. Previous studies (cf. [10, Ch.5] and references therein) have shown that the high-order compact difference method is a very efficient algorithm and has a much smaller dispersive error compared to the standard same order finite difference method.

The rest of the paper is organized as follows. In Sect. 2, we demonstrate that how the high-order compact difference scheme can be constructed for both 1D and 2D problems. Numerical results are presented in Sect. 3 to show the efficiency of our scheme. We conclude the paper in Sect. 4.

## 2. Derivation of the compact difference scheme

In the high-order compact difference methods (cf. [11, 12, 13, 14] and references therein) , the spatial derivatives in the governing PDEs are not approximated directly by the traditional explicit finite differences, but are evaluated through solving a system of linear equations. More specifically, given scalar pointwise values $u$, the derivatives of $u$ are obtained by solving a tridiagonal or pentadiagonal system. Below we will show how to develop 1D and 2D high-order compact difference schemes for solving the FitzhughNagumo equation.

**2.1. 1D compact difference scheme.** First, let us construct a sixth-order compact difference scheme to evaluate the second derivatives. Consider a uniform 1D mesh consisting of $N$ points:

$$x_1 < x_2 < \cdots < x_{i-1} < x_i < x_{i+1} < \cdots < x_N$$

with mesh size $h = x_{i+1} - x_i$. Given the function values $u_i = u(x_i), 1 \leq i \leq N$, the approximate second derivatives $u_i^{''}$ at interior points can be reconstructed by the following three point formula:

(5)

$$\alpha u_{i-1}^{''} + u_i^{''} + \alpha u_{i+1}^{''} = \frac{a}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{b}{4h^2}(u_{i+2} - 2u_i + u_{i-2}), \quad 2 \leq i \leq N-2,$$