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UNIFORMLY CONVERGENT 3-TGFEM VS LSFEM FOR SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEMS ON A SHISHKIN BASED LOGARITHMIC MESH

VIVEK SANGWAN AND B. V. RATHISH KUMAR

Abstract. In the present work, three-step Taylor Galerkin finite element method(3TGFEM) and least-squares finite element method(LSFEM) have been discussed for solving parabolic singularly perturbed problems. For singularly perturbed problems, a small parameter called singular perturbation parameter is multiplied with the highest order derivative term. As this singular perturbation parameter approaches to zero, a very sharp change occurs in the solution, which makes it difficult to find solution by traditional methods unless some special treatment is employed. A comparison on the performance of the three schemes namely, (a) 3TGFEM with exponentially fitted splines, (b) explicit least-squares finite element method with linear basis functions and (c) 3TGFEM with linear basis functions, for solving the parabolic singularly perturbed problems has been made. For all the three schemes Shishkin based logarithmic mesh has been used for numerical computations. It has been found out that the 3TGFEM scheme with exponentially fitted splines provides more accurate results as compared to the other two schemes. Detailed error estimates for the three-step Taylor Galerkin scheme with exponentially fitted splines have been presented. The scheme is shown to be conditionally uniform convergent. It is third order accurate in time variable and linear in space variable. Numerical results have been presented for all the three schemes for both linear and non-linear problems.

Key words. Boundary layer, singularly perturbed problems, mass-lumped schemes, finite element method, Taylor Galerkin method, exponentially fitted splines, least-squares method, error estimates, uniform convergence.

1. Introduction

Singularly perturbed problems appear in many areas of science and technology such as hydroelectric theory, chemical reactor theory, aerodynamics, fluid mechanics, heat transfer and problems in structural mechanics [1,2] etc. where the diffusion co-efficient can be very small as compared to the convection co-efficient. When this diffusion co-efficient becomes smaller and smaller as compared to the convection co-efficient, sharp boundary layers evolve in the solution . Because of these boundary layers, conventional methods fail to approximate the solution, particularly in the layer region. A lot of work has been done by Martin Stynes et al. [2] in this regard. We need to use very robust numerical schemes for solving the Singularly Perturbed Problems(SPP). Three-step Taylor Galerkin finite element(3TGFE) scheme is one such robust higher order numerical scheme for solving the convection dominated problems. J. Donea [3] was the first to propose the scheme for convection dominated flow problems. Because of its inherent upwinding, the scheme has been successfully employed for solving highly convective transport problems by Donea et al. [3, 4]. Later Jiang and Kawahara used the method for solving the Navier-Stokes equation [8]. As demonstrated by Peter D. Lax [5–7], performing time discretization prior to spatial discretization leads to better time accurate schemes with improved stability properties, as compared to conventional methods, 3TGFE method enjoys this benefit. The method produces particularly high phase

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accuracy and improved stability properties [4]. In this study we propose 3TGFEM for solving parabolic singularly perturbed problems. In addition to this, we have used Shishkin based logarithmic mesh in order to capture the boundary layer more sharply in the boundary layer region. Since the Bubnov Galerkin method does not provide accurate computational solution and as is suggested by Roos and Martin Stynes et al. [2,18,19] to use the exponentially fitted splines for the method to be uniformly convergent, we have also used exponentially fitted splines for the proposed 3TGFE scheme to get the added advantages of the exponentially fitted splines.

Least-squares finite element method(LSFEM) is also known for its better inherent upwinding properties. For the solution of problems in convection dominated and high-speed compressible flows, the LSFEM inherently contains a mechanism to automatically capture discontinuities, shocks or boundary layers [9]. Because of these properties and the universality, efficiency and robustness of the LSFEM, the method has been widely used for solving convection dominated flow problems, hyperbolic problems, Navier-Stokes equations and many more problems arising in the field of science and engineering [9–11].
But only a very few researchers have used LSFEM for solving SPP. Evrenosoglu and Somali approximates the solution of singularly perturbed two-point boundary value problems with LSFEM using Bezier control points[10]. In this study we also propose LSFEM for singularly perturbed parabolic differential equations.
The robustness, efficiency and accuracy of the proposed three schemes, namely, (a) 3TGFEM with exponentially fitted spline basis, (b) explicit LSFEM with linearly fitted spline basis and (c) 3TGFEM with linearly fitted spline basis have been tested both on linear and nonlinear problems using Shishkin based logarithmic mesh partitioning. Based on these numerical experiments, it has been observed that 3TGFEM with exponentially fitted splines produces more accurate results as compared to the other two schemes. Error estimates have been derived for the proposed 3TGFE scheme with exponentially fitted splines.
The organization of the paper is as follows. In Section 2, the continuous problem has been defined. Section 3 contains some results on bounds of the exact solution and its derivatives. Section 4 deals with the 3TGFEM, explicit least-squares finite element formulation and the logarithmic mesh. In Section 5, the uniform convergence of the proposed scheme has been derived and in the last Section, numerical results have been presented both for linear and nonlinear problems.

2. The Continuous Problem under Consideration

We consider the following time-dependent singularly perturbed convection-diffusion problem

$$L_\epsilon u(x,t) \equiv u_t - \epsilon u_{xx} + b(x)u_x = f(x,t),\ \forall\ (x,t) \in \Omega \tag{2.1}$$

with initial condition

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in [0,1] \tag{2.2a}$$

and boundary conditions as

$$u(0,t) = g_0(t) \quad \text{for} \quad t \in [0,1], \tag{2.2b}$$

$$u(1,t) = g_1(t) \quad \text{for} \quad t \in [0,1], \tag{2.2c}$$

where $\Omega = (0,1)^2$ and the singular perturbation parameter satisfies $0 < \epsilon \ll 1$. We assume that the convection co-efficient $b(x)$ and the source term $f(x,t)$ are