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FINITE ELEMENT ANALYSIS FOR STOKES AND NAVIER-STOKES EQUATIONS DRIVEN BY THRESHOLD SLIP BOUNDARY CONDITIONS

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Abstract. This paper is devoted to the study of finite element approximations of variational inequalities with a special nonlinearity coming from boundary conditions. After re-writing the problems in the form of variational inequalities, a fixed point strategy is used to show existence of solutions. Next we prove that the finite element approximations for the Stokes and Navier Stokes equations converge respectively to the solutions of each continuous problems. Finally, Uzawa's algorithm is formulated and convergence of the procedure is shown, and numerical validation test is achieved.

Key words. Stokes/Navier-Stokes equations, nonlinear slip boundary conditions, variational inequality, finite element method, error estimate, Uzawa's algorithm.

1. Introduction

This work is devoted to the finite element analysis of the Stokes and Navier Stokes equations driven by threshold slip boundary conditions. The Stokes systems of equations for stationary flows of incompressible Newtonian fluids we considered satisfies

$$-\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2}$$

we assume the homogeneous Dirichlet boundary condition on Γ, that is

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{3}$$

with the impermeability boundary condition

$$u_n = \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } S, \tag{4}$$

and the slip boundary condition [1, 2]

$$\left.\begin{array}{l} |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| \le g, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| < g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| = g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} \neq \boldsymbol{0}\ ,\ -(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = (g + k|\boldsymbol{u}_{\boldsymbol{\tau}}|)\dfrac{\boldsymbol{u}_{\boldsymbol{\tau}}}{|\boldsymbol{u}_{\boldsymbol{\tau}}|} \end{array}\right\} \text{ on } \quad S. \tag{5}$$

Here $\Omega \subset \mathbb{R}^d$ (d=2,3) is a bounded domain, with boundary $\partial\Omega$. It is assumed that $\partial\Omega$ is made of two components S, and Γ with $\overline{\partial\Omega} = \overline{S\cup\Gamma}$, and $S\cap\Gamma = \emptyset$. ν is a positive quantity representing the viscosity coefficient, k is the "friction" coefficient assume to be positive, and $g : S \to (0,\infty)$ is the barrier or threshold function. The velocity of the fluid is $\boldsymbol{u}$ and p stands for the pressure, while $\boldsymbol{f}$ is the external force. Furthermore, $\boldsymbol{n}$ is the outward unit normal to the boundary $\partial\Omega$ of Ω, $\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{u} - u_n\boldsymbol{n}$ is the tangential component of the velocity $\boldsymbol{u}$, and $(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = \boldsymbol{\sigma n} - (\boldsymbol{n}\cdot\boldsymbol{\sigma n})\boldsymbol{n}$ is the tangential traction. Of course, $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\nu\boldsymbol{D}(\boldsymbol{u})$ is the Cauchy stress tensor,

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where $\boldsymbol{I}$ is the identity matrix, and $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$. It should quickly be mentioned that (5) is equivalent following [3] to

$$(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} \cdot \boldsymbol{u}_{\boldsymbol{\tau}} + (g + k|\boldsymbol{u}_{\boldsymbol{\tau}}|)|\boldsymbol{u}_{\boldsymbol{\tau}}| = 0 \quad \text{on } S, \tag{6}$$

which is rewritten with the use of sub-differential as

$$-(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} \in (g + k|\boldsymbol{u}_{\boldsymbol{\tau}}|)\partial|\boldsymbol{u}_{\boldsymbol{\tau}}| \quad \text{on } S, \tag{7}$$

where the symbol $\partial|.|$ is the sub-differential of the real value function $|.|$, with $|\boldsymbol{u}|^2 = \boldsymbol{u}\cdot\boldsymbol{u}$. We recall that if X is the Hilbert space with $x_0 \in X$, then

$$y \in \partial\Psi(x_0) \Leftrightarrow \Psi(x) - \Psi(x_0) \geq y\cdot(x - x_0) \quad \text{for all} \quad x \in X\ . \tag{8}$$

The Stokes system can be considered a simplification of the Navier Stokes equations when convection is negligible. That is (1) is replaced by

$$-\nu\,\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{9}$$

with (2),(3), (4) and (5) unchanged, and the nonlinear term in (9) is the convection term given as

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \sum_{i=1}^{d} u_i \frac{\partial\boldsymbol{u}}{\partial x_i}.$$

Over the past few years a remarkable progress has been achieved in the field of computational contact mechanics. One of the key ingredients in this phenomenal growth is attributed to the better mathematical understanding of problems. The formulation by means of variational inequalities (see [3, 4, 5, 6, 7, 8, 9]) and the finite element method have contributed to the development of reliable frameworks for the numerical treatment of such problems. Despite such advances in the modeling and numerical treatment of contact problems with friction, it should be mentioned that most works reported in the literature are still restricted to solid mechanics. The numerical analysis works dealing with fluids flow are concerned with the standard Amontons-Coulomb law of perfect friction [10, 11, 12, 13, 14, 15, 16, 17, 18], replacing (5) by

$$\left.\begin{aligned} &|(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| \leq g, \\ &|(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| < g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}, \\ &|(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| = g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} \neq \boldsymbol{0}\ ,\ -(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = g\frac{\boldsymbol{u}_{\boldsymbol{\tau}}}{|\boldsymbol{u}_{\boldsymbol{\tau}}|} \end{aligned}\right\} \text{on} \quad S. \tag{10}$$

As pointed out by C. Leroux [1], such a theory can represent only a limited range of possible situations. The purpose of this work is to numerically analyze by means of finite element approximation equations (1)–(5), and (2)–(5),(9). At this juncture, it is important to recall that this type of nonlinear slip boundary conditions as far as fluid flows are concerned was first introduced by Fujita in [19, 20]. This is in continuation of a series of investigations aimed at the analysis of Stokes and Navier Stokes equations driven by nonlinear slip boundary conditions of friction type (see [10, 11, 12]). The principal goal is to analyze from the numerical analysis viewpoint the solvability, stability and convergence of the resulting variational inequalities of such problems. In order to provide a background for a better mathematical understanding of the problems, we shall introduce in Section 2 some needed tools, and quickly indicate how the problems are solvable. At this step, we recall that in C. Leroux and Tani [1, 2] a fixed point argument is used to establish the solvability of a class of problems similar to what we want to study. It is re-introduced here because of its usefulness in the finite element analysis and to make this paper self-contained.