INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 4, Number 3, Pages 224–234

A SIMPLE ANALYTIC APPROXIMATION FORMULA FOR THE BOND PRICE IN THE CHAN-KAROLYI-LONGSTAFF-SANDERS MODEL

BEÁTA STEHLÍKOVÁ

Abstract. We propose an analytic approximation formula for pricing zero-coupon bonds in the case when the short-term interest rate is driven by a one-factor mean-reverting process with a volatility proportional to the power the interest rate itself. We derive its order of accuracy. Afterwards, we suggest its use in calibration and show that it can be reduced to a simple optimization problem. To test the calibration methodology, we use the simulated data from the Cox-Ingersoll-Ross model where the exact bond prices can be computed. We show that using the approximation in the calibration recovers the parameters with a high precision.

Key words. one-factor interest rate model, Vasicek model, bond price, analytical approximation formula, order of accuracy, calibration

1. Introduction

Interest rate is a rate charged for the use of the money. As an example we show Euribor interest rates on the interbank market; Figure 1 shows the evolution of the 1-week interest rate in 2012, as well as the interest rates with different maturities (so called term structure) on a selected day.

The interest rate itself is not a tradable asset. It can be derived from the bond prices which are traded on the market. The discount bond is a secure paper which pays the unit amount of money at its maturity. A popular way of modelling bond prices is through short rate models. They are formulated in terms of a stochastic differential equation for the instantaneous interest rate (short rate). The bond prices are then given by a solution to a parabolic partial differential equation.

It is often assumed that the short rate r evolves according to the stochastic differential equation

$$dr = (\alpha + \beta r)dt + \sigma r^{\gamma} dw, \tag{1}$$

where w is a Wiener process. If the equation (1) is considered in the so called risk neutral measure, then the price $P(\tau, r)$ of the discount bond, when the current level of the short rate is r and time remaining to maturity is τ, is given by the solution to the partial differential equation

$$-\frac{\partial P}{\partial \tau} + \frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 P}{\partial r^2} + (\alpha + \beta r)\frac{\partial P}{\partial r} - rP = 0,\ r > 0,\ \tau \in (0, T), \tag{2}$$

satisfying the initial condition $P(0, r) = 1$ for all $r > 0$, see, e.g., [12], [2]. This includes the Vasicek model from [21] with $\gamma = 0$ and the Cox-Ingersoll-Ross (CIR hereafter) model proposed in [6] with $\gamma = 1/2$, in which the explicit solutions to bond pricing partial differential equations are known (cf. the original papers [21], [6] or current books on the topic of interest rate modelling, e.g., [2], [12]).

With the exception of these two models, such an explicit solution is not available. However, the empirical analysis of the market data suggests that models with

Received by the editors May 24, 2013 and, in revised form, August 14, 2013.

2000 *Mathematics Subject Classification.* 91B28, 35K15.

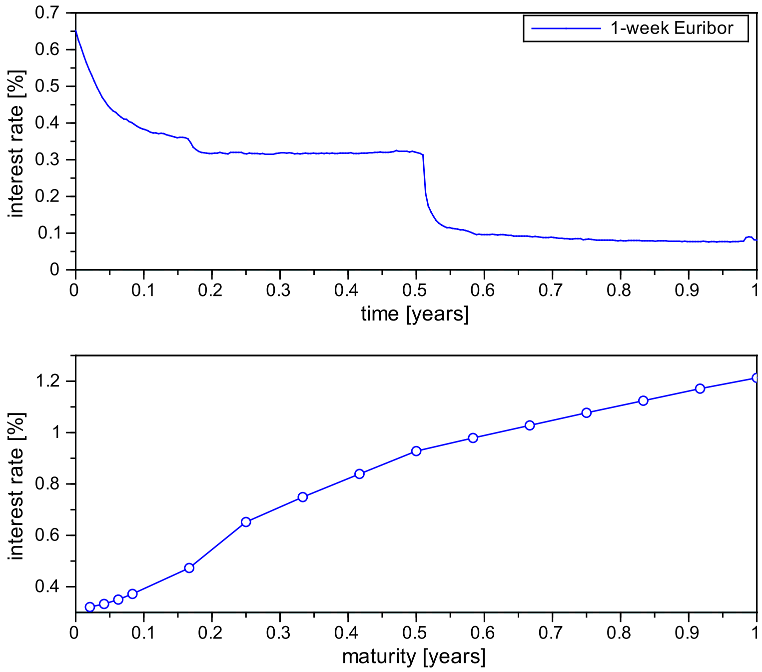

FIGURE 1. Evolution of 1-week Euribor interest rate in 2012 (above) and the term structure on 2nd July 2012 (below)

$\gamma \neq 0, 1/2$ are more suitable. The pioneering paper [3] started the discussion on the correct form of the volatility. Authors used proxy for the short rate process and estimated the parameters using the generalized method of moments. They found the parameter γ to be significantly different from the values indicated by Vasicek and CIR models. A modification of generalized method of moments (so called robust generalized method of moments), which is robust to a presence of outliers, was developed in [8]. Another contribution to this class of estimators is for example indirect robust estimation by [7]. Another popular method for parameter estimation are Nowman's Gaussian estimates [13], based on approximating the likelihood function. They were used in [9] for a wide range of interest rate markets. There are several other calibration methods for the short rate process, such as quasi maximum likelihood, maximum likelihood based on series expansion of likelihood function by Ait-Sahalia [1], Bayesian methods such as Markov chain Monte Carlo and others.

The paper [20] provides an extensive testing of the robustness of the estimation results. The authors examine the robustness over different data sets (they consider eight countries), time periods, sampling frequencies (it i important to note that the estimation procedures are based on the discretized process and the discretization error may not be negligible if the sampling interval is not sufficiently small), and estimation techniques (they consider quasi maximum likelihood and generalized method of moments). In general, the results are not robust. The highly cited result from [3] that models with $\gamma > 1$ outperform those with $\gamma \leq 1$ is not generally confirmed neither. However, the necessity to go beyond the Vasicek and CIR models is clear: using daily data (which should minimize the bias coming from discretization error) for the eight countries and both estimation methods considered, the Vasicek