INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 4, Number 2, Pages 179–202



# BIFURCATION AND SENSITIVITY ANALYSIS OF IMMUNITY DURATION IN AN EPIDEMIC MODEL

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(Communicated by Yau Shu Wong)

**Abstract.** Most disease transmission models assume no immunity or permanent immunity for simplicity, however, hosts have temporary immunity for most diseases. In this note we find that the immunity duration is actually the most sensitive parameter for dynamics of disease transmission. We provide numerical schemes to sketch Hopf bifurcations (forward or backward), sensitivity surfaces, periodicity diagrams (one dimensional or two dimensional parameter space) for a disease transmission model with immunity delay. The methods introduced here can be easily modified for a specific disease transmission model. We also test how different incidence functions change dynamics via bifurcation diagrams.

**Key words.** immunity duration, bifurcation, sensitivity, SIR, disease transmission.

## 1. Introduction

Modeling the dynamics of infectious diseases has become a topic of much interest in recent years. Such efforts are useful in disease control and in the prevention of outbreaks [1, 3]. Many types of infectious agents exist, all of which have their own unique set of behaviors. We can typically categorize infectious agents as one of the following: viral pathogens, bacterial pathogens, or parasitoids. Examples of such are as follows:

| **Viral** | **Bacterial** | **Parasitic** |
|---|---|---|
| HPV | E. Coli | Malaria |
| AIDS | Typhus | Scabies |
| SARS | Cholera | |
| Mumps | Tetanus | |
| Measles | Syphilis | |
| Influenza | Botulism | |
| Viral Hepatitis | Pink Eye | |
| Mononucleosis | Gonorrhoea | |
| Common Cold | Salmonellosis | |
| Yellow fever | Lyme Disease | |
| Small Pox | Meningitis | |
| Rubella | Typhoid | |
| Herpes | Anthrax | |
| Ebola | | |

Different infectious agents exhibit different traits and thus different dynamics arise. Parameters that depend on the specific agent include the transmission rate, the recovery rate, and the subsequent immunity duration corresponding to the specific infection. Parameters that are independent of the infectious agent typically include the natural birth and death rates of a population. Also relevant to an

Received by the editors April 14, 2013 and, in revised form, April 22, 2013.
2000 *Mathematics Subject Classification.* 37Mxx, 92Bxx, 37Nxx, 65Yxx, 65Pxx.
This research was supported by NSERC Individual Discovery Grant.

infectious agent's dynamics is an incidence function, or a function that describes how infected and susceptible individuals contact.

There are various models used to describe the dynamics of infectious diseases, some of which are as follows [3, 5]:

- SIR, or Susceptible - Infected - Recovered. SIR models assume that susceptible people become infected, recover, and then remain immune to any further infection.
- SIRS, or Susceptible - Infected - Recovered - susceptible. SIRS models assume that after a person recovers, they become susceptible again after their immunity wears off.
- SEIR, or Susceptible - Exposed - Infected - Recovered. SEIR models assume that before a person enters an infected state where the disease is contagious, they first traverse a stage of being 'exposed' meaning they carry the infectious agent, but are not able to transmit it to others.

Some models assume that a recovered person will not exhibit any sort of immunity where other models incorporate a period of immunity after recovery. The duration of immunity can be treated as constant, or in more accurate models, treated as varying. The latter incorporates the notion of distributed delay in immunity duration into functions that govern re-susceptibility.

In this note, we consider an SIRS model, with constant immunity duration, subject to parameters for the birth and death rates of the population, transmission rate of the infectious agent, and recovery rate. We use a generalized incidence function in the model, but choose a specific one for the computation of numerical results. The main goal of this note is to perform bifurcation and sensitivity analysis via numerical simulations to discuss the role of the immunity duration on the disease transmission dynamics.

## 2. Mathematical Model

We assume that a small proportion of the population has become infected by some disease. The infected individuals interact with the susceptible individuals causing more people to move from the susceptible stage to the infected stage. Infected individuals then recover at a certain rate. They remain recovered and insusceptible for a certain time while their immunity persists. After some duration of immunity, however, they become susceptible and subject to potential infection again.

We further assume that people at all stages die at a natural death rate, thus all stages decrease accordingly. The susceptible population, however, is subject to population increase due to a flux of people into the system through natural birth. We also assume that the natural birth rate and natural death rate are equal so as to maintain a constant total population.

We define variables for the population densities of the three stages, susceptible, infected, and recovered: $S(t)$ - Susceptible, $I(t)$ - Infected, $R(t)$ - Recovered/Immune. We define the parameters that represent natural mortality rate, recovery rate, transmission rate, and immunity duration: $m$ - Natural Mortality Rate, $g$ - Recovery Rate, $p$ - Transmission Rate, $\tau$ - Immunity Duration.

The model has been constructed in [2,4] in which susceptible individuals become infected at transmission rate $p$ and in proportion to the susceptible population and the incidence function f(I(t)) of the infected population. The infected individuals recover into a stage of immunity at the recovery rate $g$. After some time $\tau$, those