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MIXED FINITE ELEMENT ANALYSIS OF THERMALLY COUPLED QUASI-NEWTONIAN FLOWS

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Dedicated to Professor Walter Allegretto on the occasion of his 70th birthday

Abstract. A mixed finite element method combined with a fixed point algorithm is proposed for solving the thermally coupled quasi-Newtonian flow problem. The existence and uniqueness of the mixed variational solution are established. A more general uniqueness result for the original system problem is presented. The convergence of the approximate solution is analyzed and the corresponding error estimates are given.

Key words. Quasi-Newtonian flow, viscous heating, existence, uniqueness, nonlinear mixed method, finite element approximations, error estimates

1. Introduction

In modeling quasi-Newtonian flows with thermal effects, see for instance [4, 6, 7, 14, 17, 18, 19], we encounter a coupled system involving a quasi-Newtonian flow with a temperature dependent viscosity and a thermal balance with viscous heating. A mathematical model for this problem in two dimensions can be written as:

$$
(1.1)\quad \begin{cases}
\text{(a)} & -\boldsymbol{\nabla}\cdot(k(\theta)|D(\boldsymbol{u})|^{r-2}D(\boldsymbol{u}))+\boldsymbol{\nabla}p = \boldsymbol{f} & \text{in } \Omega\\
\text{(b)} & \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega\\
\text{(c)} & -\Delta\theta = k(\theta)|D(\boldsymbol{u})|^{r} & \text{in } \Omega\\
\text{(d)} & \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma\\
\text{(e)} & \theta = 0 & \text{on } \Gamma
\end{cases}
$$

where $\boldsymbol{u}:\Omega\to I\!R^2$ is the velocity, $p:\Omega\to I\!R$ is the pressure, $\theta:\Omega\to I\!R$ is the temperature, Ω is a bounded open subset of $I\!R^2$, Γ its boundary. The viscosity k is a function of θ, $k=k(\theta)$. $D(\boldsymbol{u})=\frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}\boldsymbol{u}^T)$ is the strain rate tensor, and $1<r<\infty$.

Professor Walter Allegretto and his former student Dr. Hong Xie did the pioneer works [1, 2, 3, 28] on the thermistor problem, which is a special scalar model with $r=2$ of the problem considered in this paper. Other works for the thermistor problem can be found in [9, 10, 11, 12, 13, 16, 21, 22], etc. In [23, 31], the complete mathematical and numerical studies such as existence, uniqueness, regularity, finite element approximations based on an iterative algorithm, convergence analysis and

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numerical implementations were presented, and then extended to the Stokes flows with viscous heating in [32]. A nonlinear finite element approximation and a mixed discontinuous Galerkin approximation were studied respectively in [8] and [34].

For the case of $r \neq 2$, the existence study for the thermally coupled nonlinear Darcy flows or Hele-Shaw flows can be found in, e.g. [7, 17, 18, 19], and [4, 29] for non-Newtonian flows with viscous heating. The existence, uniqueness, regularity, finite element approximations and convergence analysis based on the standard variational formulation for the thermally coupled nonlinear Darcy flows were studied by the second author of this paper in [30] and extended to the thermally coupled quasi-Newtonian flows in [33]. A nonlinear mixed variational formulation and finite element approximations to the thermally coupled nonlinear Darcy flows were studied recently by the authors in [35].

In this paper, we will continue the works in [33, 35] and study the nonlinear mixed variational formulation introduced in [24, 15], possessing local conservations of the momentum and the mass, for problem (1.1). We first establish the existence and uniqueness in Section 2. Because of the restriction of mathematical technique applied for nonlinear analysis as pointed out in [35], the uniqueness obtained here is for the case of $r \geq 2$ which is different from in [33], thus we get a more general result on uniqueness (see Theorem 2.3), which is another objective to study the mixed method for the nonlinear coupled problem, besides the usual one which is to get more precise numerical solution for the deviatoric stress tensor $\boldsymbol{\sigma}$. We propose a fixed point algorithm to decouple the problem in Sections 3 and its nonlinear mixed finite element approximation in Section 4. Also in Section 4, we present convergence analysis with an error estimate between continuous solution and its iterative finite element approximation.

2. Nonlinear mixed variational formulation

Let $W^{m,s}(\Omega)$ denote the Sobolev space with its norm $\|\cdot\|_{W^{m,s}}$, for $m \geq 0$ and $1 \leq s \leq \infty$. We write $H^m(\Omega) = W^{m,2}(\Omega)$ when $s = 2$, with the norm $\|\cdot\|_{H^m}$, and $L^s(\Omega) = W^{0,s}(\Omega)$ when $m = 0$, with the norm $\|\cdot\|_{L^s}$. $W_0^{m,s}(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ for the norm $\|\cdot\|_{W^{m,s}}$. Vector variables are, in general, denoted with bold face. We denote also $\boldsymbol{W}^{m,s}(\Omega) = [W^{m,s}(\Omega)]^2$, $\boldsymbol{W}_0^{m,s}(\Omega) = [W_0^{m,s}(\Omega)]^2$, $\boldsymbol{H}^m(\Omega) = [H^m(\Omega)]^2$, $\boldsymbol{H}_0^m(\Omega) = [H_0^m(\Omega)]^2$, and $\boldsymbol{L}^s(\Omega) = [L^s(\Omega)]^2$.

Throughout this work, we assume that, the coupling function μ is bounded, i.e., there exist constants $k^* \geq k_* > 0$ such that, for all $s \in I\!R$,

$$k_* \leq k(s) \leq k^* \ , \tag{2.1}$$

and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, which implies that $\boldsymbol{f} \in \boldsymbol{W}^{-1,r'}(\Omega)$, where $1/r' + 1/r = 1$, then the standard variational formulation of problem (1.1) can be defined as:

(2.2)
$$\left\{ \begin{array}{lll} \multicolumn{3}{l}{\textit{Find } (\boldsymbol{u}, p, \theta) \in \boldsymbol{W}_0^{1,r}(\Omega) \times L_0^{r'}(\Omega) \times H_0^1(\Omega) \textit{ such that}} \\ \text{(a)} & (k(\theta)|D(\boldsymbol{u})|^{r-2} D(\boldsymbol{u}), D(\boldsymbol{v})) - (p, \nabla\cdot \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), & \forall\, \boldsymbol{v} \in \boldsymbol{W}_0^{1,r}(\Omega) \\ \text{(b)} & (q, \nabla\cdot \boldsymbol{u}) = 0, & \forall\, q \in L_0^{r'}(\Omega) \\ \text{(c)} & (\nabla\theta, \nabla\eta) = (k(\theta)|D(\boldsymbol{u})|^r, \eta), & \forall\, \eta \in H_0^1(\Omega) \end{array} \right.$$

where $(\cdot,\cdot)$ denotes the dualities. $L_0^{r'}(\Omega) = \{q \in L^{r'}(\Omega) \ \Big|\ \int_\Omega q = 0\}$.