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A DIRECTION SPLITTING APPROACH FOR INCOMPRESSIBLE BRINKMAN FLOW

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Abstract. The direction splitting approach proposed earlier in [7], aiming at the efficient solution of Navier-Stokes equations, is extended and adopted here to solve the Navier-Stokes-Brinkman equations describing incompressible flows in pure fluid and in porous media. The resulting pressure equation is a perturbation of the incompressibility constraint using a direction-wise factorized operator as proposed in [7]. We prove that this approach is unconditionally stable for the unsteady Navier-Stokes-Brinkman problem. We also provide numerical illustrations of the method's accuracy and efficiency.

Key words. Direction splitting, Navier-Stokes-Brinkman equations

1. Introduction

Flows in highly porous media occur often in industrial and scientific applications. Examples are the flows through various filters (air, oil, water filters, etc.), flow of helium in pebble-bed nuclear reactors, various physiological flows like the flow in the eye of glaucoma patients, flows in mangrove swamps etc. If the porosity of the media is high, such flows are usually modelled by the Navier-Stokes-Brinkman equations. These equations include as limiting cases the Darcy model for the flow in porous media with a very low porosity, and the Navier-Stokes equations for flows with infinitely large porosity. As in the case of the classical Navier-Stokes equations, one of the major computational problems for any discretization algorithm is the imposition of the incompressibility constraint. In the case of unsteady flows probably the most popular and efficient algorithms for the imposition of incompressibility are the so-called projection methods. These methods were pioneered by Chorin [4] and Temam [14][1]. For a recent and comprehensive review on projection methods the reader is referred to [6]. All projection methods are semi-discretizations of singular perturbation of the time-dependent Stokes equations where the continuity equation is perturbed. This perturbation yields a Poisson equation for the pressure or some correction thereof with Neumann boundary condition (L^2 projection onto a divergence-free subspace of the velocity space). The solution of this Poisson equation can often be a very computationally intensive task. To circumvent this difficulty [7] proposed to use a perturbation of the continuity equation based on a direction-wise factorized operator instead of the classical Laplace operator which allows for the use of a fast tri-diagonal direct solver. In the present article we extend this approach to the case of incompressible Navier-Stokes-Brinkman flow and demonstrate numerically that it produces results of the same accuracy as the classical projection methods. We also prove that if the momentum equation is not split direction-wise, the resulting algorithm is unconditionally stable.

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[1]The authors have recently discovered that a similar velocity-pressure decoupling approach was proposed earlier in the famous article of Harlow and Welch [9] which also proposed the MAC staggered grid setting for the Stokes problem with free boundaries. Thus, we think some credit for pioneering the projection methods should be given to this article as well.

2. The fictitious domain Brinkman equations

Consider the Brinkman equations in a domain $\tilde{\Omega} \subset \mathbb{R}^d \, (d = 2, 3)$ with a Lipschitz boundary $\Gamma = \partial\tilde{\Omega}$:

$$\begin{cases} \partial_t \tilde{\mathbf{u}} - \tilde{\nu}\Delta\tilde{\mathbf{u}} + \nabla\tilde{p} + \dfrac{\tilde{\nu}}{\tilde{k}}\tilde{\mathbf{u}} = \tilde{\mathbf{f}} \quad \text{in } \tilde{\Omega} \times [0, T], \\ \nabla\cdot \tilde{\mathbf{u}} = 0 \quad \text{in } \tilde{\Omega} \times [0, T], \\ \tilde{\mathbf{u}}|_{\partial\tilde{\Omega}} = 0 \quad \text{in}[0, T], \qquad \text{and } \tilde{\mathbf{u}}|_{t=0} = \tilde{\mathbf{u}}_0 \quad \text{in} \quad \tilde{\Omega}, \end{cases} \tag{1}$$

where $\tilde{\nu}$ is the kinematic viscosity of the fluid and $\tilde{k}$ is the permeability, and T is the final time moment. In order to use the direction-splitting algorithm proposed in [7] it is necessary to extend the domain of the problem to a simple rectangle/parallelepiped (in 2D/3D). Let Ω be be such an extension i.e. $\tilde{\Omega} \subseteq \Omega$ and consider the following extension of the data

$$\nu = \tilde{\nu}, \quad \text{in} \quad \Omega, \tag{2}$$

$$\mathbf{f} = \begin{cases} \tilde{\mathbf{f}}, & \text{in} \quad \tilde{\Omega}, \\ 0, & \text{in} \quad \Omega \setminus \tilde{\Omega}, \end{cases} \tag{3}$$

$$\mathbf{u}_0 = \begin{cases} \tilde{\mathbf{u}}_0, & \text{in} \quad \tilde{\Omega}, \\ 0, & \text{in} \quad \Omega \setminus \tilde{\Omega}, \end{cases} \tag{4}$$

$$k(\mathbf{x}) = \begin{cases} \tilde{k}, & \text{in} \quad \tilde{\Omega}, \\ \nu\epsilon, & \text{in} \quad \Omega \setminus \tilde{\Omega}, \end{cases} \tag{5}$$

where $0 < \epsilon << 1$ is a penalty parameter used to enforce the boundary conditions on $\partial\tilde{\Omega}$. Then the L^2-penalty fictitious domain formulation of the problem in Ω reads

$$\begin{aligned} &\partial_t \mathbf{u}_\varepsilon - \nu\Delta\mathbf{u}_\varepsilon + \nabla p_\varepsilon + \frac{\nu}{k}\mathbf{u}_\varepsilon = \mathbf{f}, \; x \in \Omega \times [0, T] \\ &\nabla\cdot \mathbf{u}_\varepsilon = 0 \text{ in } \Omega \times [0, T], \\ &\mathbf{u}_\varepsilon|_{\partial\Omega} = 0 \text{ in } [0, T], \text{ and } \mathbf{u}_\varepsilon|_{t=0} = \mathbf{u}_0 \text{ in } \Omega. \end{aligned} \tag{6}$$

It is well known (see for example [2]) that the following result holds under sufficient regularity assumptions on the data and the domain:

$$\mathbf{u}_\varepsilon \xrightarrow[\varepsilon \to 0]{} \tilde{\mathbf{u}}, \quad \text{in } L^2(\tilde{\Omega}\times(0, T)).$$

The order of convergence depends on the regularity of the data and the domain, but it is at least $O(\epsilon^{1/2})$.

3. Numerical algorithm

3.1. Time discretization. As we mentioned in the introduction, if the domain of the problem has a simple shape, it is convenient to perturb the continuity equation as follows: $\prod_{i=1}^{d} (I - \partial_{x_i x_i})\, \phi = -\nabla\cdot\mathbf{u}/\Delta t$ where ϕ is either the pressure itself (for first order schemes) or its time increment (for higher order schemes). Therefore, it would be also convenient to apply the same direction-splitting procedure to the momentum equation. However, since the permeability is space-dependent, the direction splitting of the momentum equation, in case of an implicit treatment of the Brinkman term $\nu\mathbf{u}_\varepsilon/k$, is not straightforward. To understand the problem, let us