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# COMPUTATIONAL MODELLING OF SOME PROBLEMS OF ELASTICITY AND VISCOELASTICITY WITH APPLICATIONS TO THERMOFORMING PROCESS

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**Abstract.** The reliability of computational models of physical processes has received much attention and involves issues such as the validity of the mathematical models being used, the error in any data that the models need, and the accuracy of the numerical schemes being used. These issues are considered in the context of elastic, viscoelastic and hyperelastic deformation, when finite element approximations are applied. Goal oriented techniques using specific quantities of interest (QoI) are described for estimating discretisation and modelling errors in the hyperelastic case. The computational modelling of the rapid large inflation of hyperelastic circular sheets modelled as axisymmetric membranes is then treated, with the aim of estimating engineering QoI and their errors. Fine (involving inertia terms) and coarse (quasi-static) models of the inflation are considered. The techniques are applied to thermoforming processes where sheets are inflated into moulds to form thin-walled structures.

**Key words.** elasticity, viscoelasticity, hyperelasticity, finite element modelling, goal oriented methods, thermoforming

## 1. Introduction

The process of computational modelling for problems of continuum mechanics consists of two main phases. The mathematical model of the physics (reality) has first to be defined, after which a numerical approximation of the model has to be derived and solved to give a numerical solution in terms of quantities of interest (QoI). As each of these phases introduces error, in addition to any error in the data of the problem, the *reliability* of the process is acknowledged to be of great importance. The process of assessment of the error in the mathematical model, modelling error, is called *validation*, whilst that of the error in the numerical approximation is *verification*. Reliability is directly related to *validation* and *verification* (V & V) and is increasingly being studied; see e.g. Babuška et al. [2] and Babuška et al. [3].

In this short review paper we consider computational modelling of problems of elasticity, viscoelasticity and hyperelasticity using finite element methods. Thinking first of *verification* we present various *a priori* error analyses and *a posteriori* error estimators in the contexts of elasticity and viscoelasticity, with references to papers where these have been derived. These are followed by brief descriptions of a hyperelastic application. The *validation* of the models in this context is then addressed using goal oriented techniques as proposed by Oden and Prudhomme [4] and applied by Shaw et al. in [5].

In order to lead up to computational models for these problems, in the next section we proceed first with a framework for describing deformation and defining our notation, then address small displacement elasticity and viscoelasticity, and

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finally progress to hyperelastic (large) deformation. The last section of the paper deals with the computational modelling of thermoforming processes.

## 2. Mathematical models, weak formulations and finite element methods

**2.1. Solid Mechanics Framework (Small Displacement Case).** Let $\mathcal{G}$ be a compressible solid body with mass density $\rho$ which in its undeformed state occupies the open bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2,3$ with polygonal/polyhedral boundary $\partial\Omega$. A point in $\bar{\Omega} = \Omega \bigcup \partial\Omega$ is denoted by $\mathbf{x} \equiv (x_i)_{i=1}^3$, when $n = 3$. The boundary $\partial\Omega$ is partitioned into disjoint subsets $\Gamma_D$ and $\Gamma_N$ such that $\partial\Omega \equiv \Gamma_D \bigcup \Gamma_N$, $\Gamma_D \bigcap \Gamma_N = \emptyset$ and meas $(\Gamma_D) > 0$. Suppose that, for time $t \in I \equiv (0, T]$, $T > 0$, the body $\mathcal{G}$ is acted upon by body forces

$$\mathbf{f}(\mathbf{x}, t) \equiv (f_i(\mathbf{x}, t))_{i=1}^3 ,$$

for $\mathbf{x} \in \Omega$ and surface tractions

$$\mathbf{g}(\mathbf{x}, t) \equiv (g_i(\mathbf{x}, t))_{i=1}^3 ,$$

for $\mathbf{x} \in \Gamma_N$. The displacement at a point $\mathbf{x}$ under the action of the forces $\mathbf{f}$ and $\mathbf{g}$ is $\mathbf{u} \equiv (u_i(\mathbf{x}, t))_{i=1}^3$, $\mathbf{x} \in \Omega$, $t \in I$, and with a small displacement assumption $\mathbf{x} + \mathbf{u} \approx \mathbf{x}$, so that we do not need to distinguish between the deformed and undeformed domains in most terms. Let $\underline{\sigma} \equiv (\sigma_{ij})_{i,j=1}^3 \equiv (\sigma_{ij}(\mathbf{x}, t))_{i,j=1}^3$ denote the stress resulting from the deformation.

Applying Newton's second law of motion, relating force to the rate of change of linear momentum, to this configuration we obtain the momentum equations

$$\rho(\mathbf{x})\ddot{u}_i(\mathbf{x}, t) - \sigma_{ij,j}(\mathbf{x}, t) = f_i(\mathbf{x}, t), \quad i = 1,2,3 \text{ in } \Omega \times I. \tag{1}$$

and these together with the boundary and initial conditions

$$u_i(\mathbf{x}, t) = 0 \quad \text{in } \Gamma_D \times \bar{I}, \tag{2}$$

$$\sigma_{ij}\hat{n}_j = g_i(\mathbf{x}, t), \quad \text{in } \Gamma_N \times \bar{I}, \tag{3}$$

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{4}$$

$$\dot{u}_i(\mathbf{x}, 0) = u_i^1(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{5}$$

define the dynamic deformation problem, where $\hat{\mathbf{n}} \equiv (\hat{n}_i)_{i=1}^n$ is the unit outward normal to $\Gamma_N$, the Einstein convention has been used, and $v_{,j} \equiv \partial v / \partial x_j$.

If the inertia terms can be neglected in the deformation and assuming that $\mathbf{u}(\mathbf{x}, t) = 0 \ \forall\, t < 0$, we obtain the quasistatic problem, where $i, j = 1,2,3$,

$$-\sigma_{ij,j}(\mathbf{x}, t) = f_i(\mathbf{x}, t), \text{ in } \Omega \times I \tag{6}$$

$$u(\mathbf{x}, t) = 0, \quad \text{in } \Gamma_D \times \bar{I} \tag{7}$$

$$\sigma_{ij}\hat{n}_j = g_i(\mathbf{x}, t), \quad \text{in } \Gamma_N \times \bar{I}, \tag{8}$$

In order to complete the definitions of the dynamic and quasistatic problems it is necessary to have a constitutive relationship connecting the stress to the displacement and its derivatives. The constitutive relationship reflects the behaviour of the material of the body $\mathcal{G}$.