INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 3, Number 3, Pages 297–306

A COMPARATIVE STUDY OF THE EFFICIENCY OF JET SCHEMES

PRINCE CHIDYAGWAI, JEAN-CHRISTOPHE NAVE, RODOLFO RUBEN ROSALES, AND BENJAMIN SEIBOLD

Abstract. We present two versions of third order accurate jet schemes, which achieve high order accuracy by tracking derivative information of the solution along characteristic curves. For a benchmark linear advection problem, the efficiency of jet schemes is compared with WENO and Discontinuous Galerkin methods of the same order. Moreover, the performance of various schemes in tracking solution contours is investigated. It is demonstrated that jet schemes possess the simplicity and speed of WENO schemes, while showing several of the advantages as well as the accuracy of DG methods.

Key words. jet scheme, advection, high-order, WENO, DG, efficiency, contours.

1. Introduction

The advection of field quantities under a velocity field is an important sub-problem in many computational projects. Examples are the passive transport of concentrations, or the movement of interfaces using level set approaches [13]. We consider the linear advection equation

$$\phi_t + \vec{v} \cdot \nabla \phi = 0 \ , \tag{1}$$

which moves a scalar quantity $\phi(\vec{x}, t)$ by a given velocity field $\vec{v}(\vec{x}, t)$. Equation (1) is augmented with initial conditions $\phi(\vec{x}, 0) = \Phi(\vec{x})$ and boundary conditions. All data is assumed smooth in space and time. Furthermore, it is assumed that the problem at hand can be described by, or embedded into, a rectangular computational domain, equipped with a regular grid.

High order accurate numerical approximations of (1) on a fixed grid are commonly based on schemes that employ polynomials of sufficient degree to interpolate smooth solutions with the required accuracy. One popular type of approach are finite difference ENO [16] or WENO [11] methods. These store approximations of the solution values at the grid points, and achieve a high order polynomial approximation by considering local neighborhoods that are several grid points wide. Another type of approach are discontinuous Galerkin (DG) [14, 4, 5] methods. These achieve high order accuracy by storing a high degree polynomial approximation in each grid cell, and approximating the flux through cell boundaries based on a weak formulation of (1). Both WENO and DG methods are based on a semi-discretization of (1), and achieve high accuracy in time by using strong stability preserving (SSP) Runge-Kutta schemes [16, 6, 7].

Both types of approaches incur problems and difficulties. Due to the wide stencils of WENO methods, it is challenging to preserve high order near boundaries. Furthermore, their non-locality poses difficulties for an effective parallelization, and their use in conjunction with adaptive grid approaches [2]. In contrast, in DG methods, communication is limited to neighboring cells. However, DG approaches are characterized by costly quadratures over grid cells and their edges, significant time step restrictions, and non-trivial implementation aspects.

A new class of approaches for (1), so called *jet schemes*, has been proposed recently [12, 15]. The goal of these methods is to provide an attractive compromise between WENO and DG methods, possessing the optimal locality and good resolution properties of DG, while being close to the computational efficiency and ease of implementation of WENO schemes. In this paper we conduct a comparative study of the efficiency and accuracy of this new class of methods. We consider two versions of third order accurate jet schemes, and compare them with WENO and DG methods of the same order. The considered jet schemes are described in Sect. 2. Then, in Sect. 3, DG methods are outlined, and in Sect. 4, information about the WENO schemes that we employ is given. Sect. 5 shows the numerical results obtained when applying the three types of approaches to

Received by the editors February 2011 and, in revised form, March 2012 .
2000 *Mathematics Subject Classification.* 65M25; 65M12; 35L04.

two versions of a benchmark problem. In Sect. 6, a discussion of the observations and theoretical estimates of the computational cost are given.

2. Jet Schemes

Jet schemes are based on an advect–and–project approach in function spaces. Given an approximation to the solution of (1) at time t_n, an approximate solution at time $t_{n+1} = t_n + \Delta t$ is obtained by the time step

$$\phi^{n+1} = P \circ A_{t_{n+1},t_n} \phi^n \ .$$

Here A_{t_{n+1},t_n} is an approximate advection operator, defined by evolving the solution along characteristics using a numerical ODE solver, and P is a projection operator, given by a piecewise Hermite interpolation based on parts of the jet of the solution at the points of a cartesian grid. The fact that the projection requires data only at grid points allows to use this approach as a numerical scheme: when evaluating the advection operator, only the characteristic curves that go through grid points at t_{n+1} need to be considered. In addition, the evolution of derivatives of the solution along these characteristic must be found. As outlined below, the required spacial derivatives of the advection operator can be found by analytical differentiation (Sect. 2.3) or by approximations based on tracking multiple nearby characteristics (Sect. 2.5).

Jet schemes can be constructed in any space dimension, and for any order of accuracy [15]. For simplicity, here we only describe third order schemes in two space dimensions. We consider a rectangular computational domain $\Omega \subset \mathbb{R}^2$, equipped with a regular cartesian grid of grid size h.

2.1. Projection. In a grid cell $[a, a+h] \times [b, b+h] \subset \Omega$, let the vertices be indexed by a vector $\vec{q} \in \{0,1\}^2$, such that the vertex of index $\vec{q}$ is at position $\vec{x}_{\vec{q}} = (a + h\,q_1, b + h\,q_2)$. On each of the four vertices, let a vector of data be given by $\phi_{\vec{\alpha}}^{\vec{q}} \ \forall\, \vec{\alpha} \in \{0,1\}^2$. The data represents partial derivatives of orders up to 1 in each variable, as follows: $\vec{\alpha} = (0,0)$ represents function values ϕ; $\vec{\alpha} = (1,0)$ and $\vec{\alpha} = (0,1)$ represent first derivatives $\partial_x \phi$ and $\partial_y \phi$, respectively; and $\vec{\alpha} = (1,1)$ represents $\partial_{xy} \phi$. This data is interpolated by the bi-cubic polynomial

$$\mathcal{H}(\vec{x}) = \sum_{\vec{q},\vec{\alpha} \in \{0,1\}^p} \phi_{\vec{\alpha}}^{\vec{q}}\ W_{\vec{\alpha}}^{\vec{q}}(\vec{x}) \ , \tag{2}$$

where the $W_{\vec{\alpha}}^{\vec{q}}(\vec{x})$ are bi-cubic basis functions, given by the tensor product formulas

$$W_{\vec{\alpha}}^{\vec{q}}(\vec{x}) = h^{\alpha_1+\alpha_2}\ w_{\alpha_1}^{q_1}\left(\tfrac{x-a}{h}\right) w_{\alpha_2}^{q_2}\left(\tfrac{y-b}{h}\right) \ ,$$

and the w_α^q are the univariate basis functions

$$w_0^0(x) = 1 - 3x^2 + 2x^3, \quad w_0^1(x) = 3x^2 - 2x^3, \quad w_1^0(x) = x - 2x^2 + x^3, \quad w_1^1(x) = -x^2 + x^3.$$

The bi-cubic interpolant (2) is an $O(h^4)$ accurate approximation to any sufficiently smooth function ϕ that it interpolates on a cell of size h [15].

On the computational domain $\Omega \subset \mathbb{R}^2$, let the grid points be labeled $\vec{x}_{\vec{m}}$, where $\vec{m} \in \mathbb{Z}^2$. For any (sufficiently smooth) function $\phi : \Omega \to \mathbb{R}$, we define a global interpolant $\mathcal{H}_\phi$ as follows: at each grid point $\vec{x}_{\vec{m}}$, evaluate the derivatives of ϕ, $\partial_x \phi$, $\partial_y \phi$, and $\partial_{xy} \phi$, to produce a data vector $\phi_{\vec{\alpha}}^{\vec{m}} \ \forall\, \vec{\alpha} \in \{0,1\}^2$. Then, on each grid cell, use this data to define the bi-cubic interpolant (2). In the function space

$$S^{2,+} = \{\psi \in C^1 : \psi \text{ twice differentiable a.e. with } D^2\psi \in L^\infty\} \ ,$$

this procedure can be applied using the following convention: whenever $\partial_{xy}\psi$ must be evaluated at a point at which $D^2\psi$ is not defined in the classical sense, we define it as

$$\partial_{xy}\psi(\vec{x}_{\vec{m}}) = \tfrac{1}{2}\Big(\operatorname*{ess\,lim\,sup}_{\vec{x}\to\vec{x}_{\vec{m}}} \partial_{xy}\psi(\vec{x}) + \operatorname*{ess\,lim\,inf}_{\vec{x}\to\vec{x}_{\vec{m}}} \partial_{xy}\psi(\vec{x})\Big) \ .$$

The view on the general order of approximation case [15] reveals the rationale for the notation $S^{2,+}$. Clearly, the re-application of the interpolation procedure does not change the result: $\mathcal{H}_{\mathcal{H}_\phi} = \mathcal{H}_\phi$. Hence, in the space $S^{2,+}$ it can be formulated as a projection operator

$$P\phi = \mathcal{H}_\phi \ . \tag{3}$$