INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 3, Number 3, Pages 285–296

INVERSE TEMPERATURE RECONSTRUCTION IN THERMOCAPILLARY-DRIVEN THIN LIQUID FILMS

MATHIEU SELLIER AND SATYANANDA PANDA

Abstract. A thin liquid film subject to a temperature gradient undergoes thermocapillary convection because of the non-uniform surface tension at the free surface. This induced flow perturbs the film free surface and generate a free surface velocity field. These observable consequences can be thought of as the "signature" of the imposed temperature field and this work investigates whether the temperature field can be reconstructed from this signature for general three-dimensional flows. Using a model based on the lubrication approximation, we show that one can explicitly formulate the partial differential equation which governs this inverse problem. This equation is solved using finite differences. We illustrates the feasibility of this reconstruction exercise on a set of "artificial" experimental data obtained by first solving the direct problem which consists in computing the free surface deformation and free surface velocity field for a given applied temperature field.

Key words. Inverse problem, Thin liquid film, Thermocapillary.

1. Introduction

Consider an oil liquid layer heated in a pan on a stove. The non-uniform heating induces an inhomogeneous temperature field in the oil film resulting in a surface tension gradient at the oil free surface. This surface tension gradient generates a flow at the free surface and throughout the layer as a result of viscous drag. This well-known phenomena, first discussed by Benard [1] early in the 20^{th} century, is commonly referred to as the thermocapillary effect. Since this early observation of the phenomena, the problem has received considerable attention because of its importance in a range of industrial applications such as the growth of crystals in semiconductor materials, the rupture of thin films in heat transfer devices, or the texturing of surfaces in magnetic storage devices. A comprehensive review of the literature pertaining to this phenomenon can be found in reference [2]. A consequence of this flow induced by temperature variations is that the free surface deforms and a velocity field develops at the free surface of the film. We can therefore think of these phenomena as a signature of the imposed temperature field and a natural question to consider is whether and how one can reconstruct the temperature field from the knowledge of the local film thickness or the free surface velocity field. This could potentially provide an alternative strategy to estimate the heat transfer on surface coated by a liquid layer.

A number of techniques have been proposed in the literature to measure liquid film thickness. We can mention here the needle contact technique as used by Burelbach et al. or Koehler et al. [3, 4], which provides point-wise values of the film thickness. Alternatively, optical techniques are routinely used to generate maps of free surface deformations. Such techniques include the shadowgraph technique [4, 5] or interferometry [6, 7], to name but a few. The interested reader is referred

Received by the editors November, 2011 and, in revised form, March, 2012 .

2000 *Mathematics Subject Classification.* 76T10, 65L09.

Part of this work was completed while on a visit to the Oxford Centre for Collaborative Applied Mathematics (OCCAM). MS gratefully acknowledges the financial support of OCCAM.

to the book of de Gennes et al. [8] for a good review of the film thickness measurement techniques. Arguably, measuring the free surface velocity can be done more readily by introducing tracers in the fluid and using Particle Image Velocimetry to reconstruct the velocity field. For example, this technique was successfully used by Kanemura et al. [9] as a diagnostic system to monitor Lithium flow at the International Fusion Materials Irradiation Facility in Japan and by Eswaran et al. [10] to monitor the the liquid free surface velocity in a moving tank.

The present authors have recently considered this problem for planar flows, i.e. the temperature field and film thickness only vary in one direction [11]. By adopting the lubrication approximation which considerably simplifies the description of the problem, the authors derived a closed-form solution to this problem in the steady case. Given the strongly nonlinear nature of the fourth-order partial differential equation (pde) expressing the lubrication approximation, the existence of a closed-form solution to the inverse problem appears quite fortunate. Such inverse problems are typically solved in the pde-constrained optimization framework [12] which involves the following steps:

- Parametrize the unknown input (the temperature field at the solid surface in the present case);
- Defines an objective function measuring the mismatch between the measured data and the computed ones (the difference between the measured and computed free surface profiles in the present case);
- Performs a sensitivity analysis to identify a direction in the parameter space which results in a decrease of the objective functions;
- Update the input accordingly.

This well tested approach has the disadvantage of being difficult to implement and the formulation of the sensitivity is a particularly involved stage. The procedure we describe here is a one step approach since, for this particular problem, we are able to reformulate the governing equations in such a way that we can explicitly derive a pde which governs the inverse problem. The present authors adopted an analogous approach to infer an unknown substrate topography from the knowledge of the free surface deformation [13, 14]. This approach was later extended by Heining to include the effect on inertia [15] and reconstruct the velocity field in the film [16] in addition to the substrate topography.

The paper is organized as follows; the next two sections describe the governing equation in continuous and discretized forms and the solution procedure. A section with numerical results for the direct and inverse problems follows. The paper closes with a discussion and conclusions.

2. The direct problem

2.1. The field equations. The problem we consider here is illustrated in Figure 1. A thin liquid layer of characteristic thickness H_0 rests on a square solid substrate $(2\pi L \times 2\pi L)$ heated at temperature $T_s(X, Y)$ where $(X,\ Y)$ are the spatial coordinates attached to the solid surface. The liquid, assumed to be Newtonian and incompressible, has viscosity μ, density ρ, specific heat c_p, thermal conductivity k, and the local surface tension σ depends on the temperature T according to

$$\sigma = \sigma_0 - \gamma_0 \left(T - T_0\right), \tag{1}$$

where σ_0 is the surface tension at the reference temperature T_0 and γ_0 is a constant. The liquid layer exchange heat with the surrounding by convective heat transfer.