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COALESCENCE CUBIC SPLINE FRACTAL INTERPOLATION SURFACES

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Abstract. Fractal geometry provides a new insight to the approximation and modelling of scientific data.This paper presents the construction of coalescence cubic spline fractal interpolation surfaces over a rectangular grid D through the corresponding univariate basis of coalescence cubic fractal splines of Type-I or Type-II. Coalescence cubic spline fractal surfaces are self-affine or non-self-affine in nature depending on the iterated function systems parameters of these univariate fractal splines. Upper bounds of L_∞-norm of the errors between between a coalescence cubic spline fractal surface and an original function $f \in C^4[D]$, and their derivatives are deduced. Finally, the effects of free variables, constrained free variables and hidden variables are discussed for coalescence cubic spline fractal interpolation surfaces through suitably chosen examples.

Key words. Fractals, Iterated Function System, Fractal Interpolation Surface, Cardinal Cubic Spline, Hidden Variables, CHFIS, Non-self-affine and Surface Approximation

1. Introduction

The theory of fractal interpolation has become a powerful tool in applied science and engineering since Barnsley [1] introduced fractal interpolation function (FIF) using the theory of iterated function system (IFS). The attractor of an IFS is the graph of a FIF that interpolates a given set of data points. Fractal interpolation constitutes an advance in techniques of approximation in the sense that these functions used are not necessarily differentiable, and show the rough aspect of real-world signals [2–4]. For smooth curve approximation through fractal methodology, Barnsley and Harrington [5] initiated the construction of a differentiable FIF or C^r-FIF f that interpolates the prescribed data if values of $f^{(k)}$, $k = 1, 2, \ldots, r$, are assigned at the initial end point of the interval. Fractal splines with general boundary conditions are studied recently [6, 7]. The power of fractal methodology allows us to generalize almost any other interpolation techniques, see for instance [8, 9].

Fractal surfaces are proved to be useful to approximate various type of surfaces in material science, ocean engineering, geology, chemistry, physics, image processing and computer graphics. Massopust [10] was first to put forward the construction of fractal interpolation surfaces (FISs) wherein he assumed the surface as triangular simplex and interpolation points on the boundary to be co-planar. In view of lack of flexibility in this construction, Geronimo and Hardin [11] and Zhao [12] have generalized the construction of FIS by allowing more general boundary data. Xie and Sun [13] constructed bivariate FIS on rectangular grids with arbitrary contraction factors and without any condition on boundary points. Dalla [14] extemporised this construction by using collinear boundary points and demonstrated that the attractor is a continuous FIS. Further research and developments on FISs in various directions are discussed by Massopust [15], Bouboulis, et al. [16, 17], Chand and Navascués [18] , Metzler and Yunb [19]. However, all the constructions mentioned above lead to self-affine fractal surfaces.

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The term 'hidden variable' was introduced by Barnsley et al. [20] and Massopust [21]. A hidden variable FIF (HFIF) is more diverse, appealing and irregular than a FIF for the same set of interpolation data as functional values of a HFIF continuously depend on all the defining IFS parameters. Since a HFIF is the projection of a vector valued function, it is usually non-self-affine in nature. Bouboulis and Dalla [22] have constructed hidden variable vector valued FIFs on random grids in $\mathbb{R}^2$. Chand and Kapoor [23] have introduced the coalescence hidden variable FIF and studied their stability analysis [24]. A non-diagonal IFS that generates both self-affine and non-self-affine FIS simultaneously, depending on the free variables and constrained variables on a general set of interpolation data is constructed in [25]. The attractor of such an IFS is called the coalescence hidden-variable fractal interpolation surface (CHFIS). A CHFIS is a preferred choice for the study of highly uneven surfaces such as clouds, sea surfaces, surfaces of rocks, tsunami waves, etc. The quantification of smoothness of such surfaces in terms of Lipschitz exponent of its corresponding CHFIS is investigated recently in [26]. This paper aims to develop the theory of coalescence cubic spline FISs (CCFISs), to study their convergence results, and to validate the effects of IFS parameters on the shape of a CCFIS.

In Section 2, we discuss basics of coalescence hidden variable FIFs (CHFIFs), construction of cubic spline CHFIFs and cardinal cubic CHFIFs. An estimate of the error bound of the cubic spline CHFIF with the original function is obtained in this section. The construction of CCFISs is carried out in Section 3 through tensor product of cardinal cubic spline CHFIFs of Type-I or Type-II. Upper bounds of L_∞-norm of the errors between a cubic spline CHFIF and the original function $f \in C^4[D]$, and their derivatives are deduced in Section 4. Finally, the effects of IFS parameters on a CCFIS are illustrated through various suitably chosen examples.

2. Cubic Spline CHFIFs

We discuss the basics of CHFIFs through IFS theory in Section 2.1. The construction of cubic spline CHFIFs and cardinal cubic spline CHFIFs of Type-I or Type-II are described respectively in Section 2.2 and Section 2.3. Upper bounds of L_∞-norm of the errors between a cubic spline CHFIF and an original function, and their derivatives are estimated in Section 2.4.

2.1. Basics of CHFIFs. Let $\Delta_t : t_0 < t_1 < \cdots < t_N$ be a partition of an interval $I = [t_0, t_N] \subset \mathbb{R}$ and $\{(t_j, x_j) \in I \times \mathbb{R} : j = 0, 1, 2, \ldots, N\}$ be a set of interpolation data points. This data set is extended to a generalized set of data $\{(t_j, x_j, \xi_j) \in \mathbb{R}^3 : j = 0, 1, 2, \ldots, N\}$ with real parameters $\xi_j,\ j = 0, 1, 2, \ldots, N$. Let $L_j : I \longrightarrow I_j = [t_{j-1}, t_j]$ be a contraction map satisfying

$$L_j(t_0) = t_{j-1}, \quad L_j(t_N) = t_j \text{ for } j = 1, 2, \ldots, N. \tag{2.1}$$

Let $F_j : I \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a vector valued function satisfying

$$\begin{cases} F_j(t_0, x_0, \xi_0) = (x_{j-1}, \xi_{j-1}),\ F_j(t_N, x_N, \xi_N) = (x_j, \xi_j), \\ \quad d(F_j(t, x, \xi), F_j(t^*, x^*, \xi^*)) \le \tau_j d_E((x, \xi), (x^*, \xi^*)), \end{cases} \tag{2.2}$$

for $j = 1, 2, \ldots, N$, where $(t, x, \xi),\ (t^*, x^*, \xi^*) \in I \times \mathbb{R}^2$, $0 \le \tau_j < 1$, d is the sup. metric on $I \times \mathbb{R}^2$, and d_E is the Euclidean metric on $\mathbb{R}^2$. In order to define a CHFIF, functions L_j and F_j are chosen such that $L_j(t) = a_j t + b_j$ and

$$F_j(t, x, \xi) = A_j(x, \xi)^T + (p_j(t), q_j(t))^T \equiv (F_j^1(t, x, \xi), F_j^2(t, \xi))^T, \tag{2.3}$$