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SCHEME OF THE SOLUTION DECOMPOSITION METHOD FOR A SINGULARLY PERTURBED REACTION-DIFFUSION EQUATION; APPROXIMATION OF SOLUTIONS AND DERIVATIVES

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Abstract. A Dirichlet problem is considered for a singularly perturbed ordinary differential reaction-diffusion equation. For this problem, a new approach is developed to construct difference schemes convergent uniformly with respect to a perturbation parameter ε for $\varepsilon \in (0, 1]$, i.e., ε-uniformly. This approach is based on a *decomposition of the discrete solution* into the regular and singular components which are solutions of *discrete subproblems* considered on *uniform grids*. Using an *asymptotic construction technique*, a *difference scheme* of the *solution decomposition method* is constructed that converges ε-uniformly in the maximum norm at the rate $\mathcal{O}\left(N^{-2} \ln^2 N\right)$, where $N + 1$ is the number of nodes in the grids used; for fixed values of the parameter ε, the scheme converges at the rate $\mathcal{O}\left(N^{-2}\right)$. For the constructed scheme, approximations of the regular and singular components to the solution and their derivatives up to the second order are studied. A modified scheme of the solution decomposition method is constructed for which the regular component of the solution and its discrete derivatives converge ε-uniformly in the maximum norm at the rate $\mathcal{O}\left(N^{-2}\right)$ for $\varepsilon = o(\ln^{-1} N)$.

Key words. singularly perturbed boundary value problem, ordinary differential reaction-diffusion equation, discrete solution decomposition, asymptotic construction technique, difference scheme of a solution decomposition method, uniform grids, ε-uniform convergence, maximum norm, approximation of derivatives.

1. Introduction

At present, for singularly perturbed boundary value problems, methods for constructing ε-uniformly convergent difference schemes *on grids condensing in a neighborhood of the boundary layer* (using the classical schemes on the Bakhvalov and/or Shishkin grids) are well developed. Such methods have widespread application due to their simplicity and convenience (see, e.g., [1]–[6] and the bibliography therein). A drawback of these numerical methods is the necessity to solve discrete equations on grids in which step-sizes change sharply in a neighborhood of the boundary layer. Such type grids do not allow one to apply high-efficiency computational methods developed for solving grid equations in the case of regular boundary value problems (see, e.g., [7, 8]), and also give rise to difficulties in the construction of high-order accurate schemes (see, e.g., [9, 10]) and in approximation of derivatives.

Note that fitted operator methods (see the description in [11]–[13], and also [3], Ch. 4; [4], Ch. 2; [5], Part I, Ch. 4, and the bibliography therein) have advantage in the simplicity of uniform grids used; however, the coefficients of grid equations in these methods depend on the explicit form of the main term in the singular component of the solution. For this reason, fitted operator methods have a restricted applicability in constructing ε-uniformly convergent numerical methods, in particular, for problems with parabolic initial or boundary layers (see [14, 15] and also [2], Ch. II, § 1; [4], Ch. 6; [6], Ch. 1, 9 and the bibliography therein).

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In the present paper, for a model singularly perturbed boundary value problem for an ordinary differential reaction-diffusion equation, a *new approach* based on the asymptotic construction technique is proposed to construct special schemes, namely, the *method of decomposition of a grid solution* using the *classical approximations of subproblems* on *uniform grids* for the regular and singular components of the solution. Unlike known approaches to the construction of ε-uniformly convergent difference schemes — the *fitted operator method* and the *method of special condensing grids* — discrete subproblems *in the new approach* are solved on *uniform grids*; moreover, the *coefficients* of grid equations *do not depend on the explicit form of the singular component* of the solution. Construction of difference schemes of the new solution decomposition method have been developed in the papers [16]–[18], where approximation of the solution have been studied only. In the present paper, approximations of the regular and singular components of the solution and their derivatives up to the second order are examined for schemes based on the solution decomposition method. Earlier, such a problem was not considered.

Contents of the paper. The formulation of the boundary value problem for a singularly perturbed reaction-diffusion equation and the aim of the research are presented in Section 2. Standard difference schemes on uniform and piecewise-uniform grids are given in Section 3. A difference scheme of the solution decomposition method is constructed in Section 4. This approach applies the decomposition of a grid solution into its regular and singular component considered on uniform grids. The scheme of the solution decomposition method converges ε-uniformly at the rate $\mathcal{O}\left(N^{-2}\ln^2 N\right)$, which is the same as for the standard scheme on a piecewise uniform grid. Here, $N+1$ is the number of nodes in the grids used; note that the number of nodes is the same in the grids on which grid approximations of the regular component, as well as the singular components, are considered in neighborhoods of the left and right boundaries of the domain. Approximations of the regular and singular components of the solution and their derivatives are discussed in Section 5. "Standard" a priori estimates used in the constructions are considered in Section 6. A modified scheme of the solution decomposition method for which the regular component of the solution and its discrete derivatives converge ε-uniformly in the maximum norm at the rate $\mathcal{O}\left(N^{-2}\right)$ for $\varepsilon = o(\ln^{-1} N)$ is constructed in Section 7.

2. Problem Formulation. Aim of Research

On the set $\overline{D}$

$$\overline{D} = D \bigcup \Gamma, \quad D = (0,d), \quad \Gamma = \Gamma_1 \cup \Gamma_2, \tag{2.1}$$

where Γ_1 and Γ_2 are the left and right parts of the boundary Γ, we consider the Dirichlet problem for the ordinary differential reaction-diffusion equation

$$L\,u(x) \equiv \left\{\varepsilon^2\, a(x)\frac{d^2}{dx^2} - c(x)\right\} u(x) = f(x), \quad x \in D, \quad u(x) = \varphi(x), \quad x \in \Gamma. \tag{2.2}$$

The functions $a(x)$, $c(x)$, $f(x)$ are assumed to be sufficiently smooth on $\overline{D}$; moreover,[†]

$$a_0 \le a(x) \le a^0, \quad c_0 \le c(x) \le c^0, \quad x \in \overline{D}; \quad a_0,\, c_0 > 0; \tag{2.3}$$
$$|f(x)| \le M, \quad x \in \overline{D}; \qquad |\varphi(x)| \le M, \quad x \in \Gamma;$$

[†] We denote by M (by m) sufficiently large (small) positive constants that do not depend on the value of the parameter ε. In the case of grid problems, these constants are also independent of the stencils of the difference schemes.