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ANALYSIS AND NUMERICAL METHODS FOR SOME CRACK PROBLEMS

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Abstract. In this paper, finite difference schemes based on asymptotic analysis and the augmented immersed interface method are proposed for potential problems with an inclusion whose characteristic width is much smaller than the characteristic length in one and two dimensions. We call such a problem as a crack problem for simplicity. In the proposed methods, we use asymptotic analysis to approximate the problem with a single sharp interface. The jump conditions for the interface problem are derived. For one-dimensional problem, or two-dimensional problems in which the center line of the crack is parallel to one of axis, we can simply modify the finite difference scheme with added correction terms at irregular grid points. The coefficient matrix of the finite difference equations is still an M-matrix. For problems with a general thin crack, an augmented variable along the center line of the crack is introduced so that we can apply the immersed interface method to get the discretization. The augmented equation is the asymptotic jump condition. Numerical experiments including the case with large jump discontinuity in the coefficient are presented.

Key words. Crack problem, open-ended interface, asymptotic analysis, jump conditions, Cartesian grid method, augmented immersed interface method.

1. Introduction

Many practical application problems involved open-ended interfaces, cracks, and thin surfaces, for example, the motion of a flag or wing of a butterfly. It is challenging to solve those problems that can capture the physical behaviors near the open-ended interfaces or cracks. In this paper, we consider such a problem in the potential theory,

$$-\nabla \cdot (\beta(x)\nabla u(x)) = 0, \tag{1}$$

in a domain Ω. We assume that within Ω, there is an inclusion whose characteristic width is much smaller than its characteristic length in one and two dimensions, see Figure 1 for an illustration. We call such a problem as a crack problem for short even though a real crack problem is much more complicated. Our new method is based on asymptotic analysis, augmented strategies, and modified finite difference equations.

One difficulty in solving a crack problem is that the width of a crack may be so small that there is either no or very few grid points inside the crack. Nevertheless, the potentials inside and outside the crack may be significantly different. Numerical methods based on adaptive meshes can be applied to solve such problems. But the method may be complicated and can not utilize fast Poisson solvers. Our goal in this paper is to develop simple Cartesian grid methods to solve the crack problems.

We were first introduced with the crack problem by some scientist from Schlumberger company in Ridgefield, Connecticut, USA. The initial idea of combining the asymptotic analysis and the augmented finite difference method was proposed by Z. Li during the scientific meeting [6]. In [17], the authors have derived the asymptotic relations of the crack problems. The results in [17] have been basis for the

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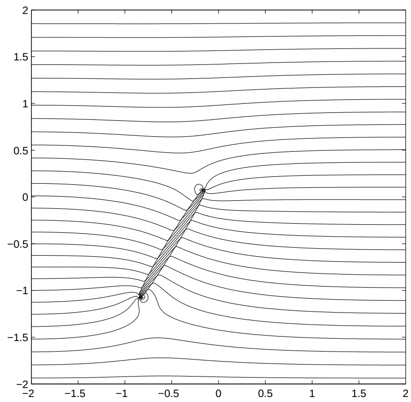

FIGURE 1. A plot of the computed potential of a crack (a thin ellipse). The coefficient is $\beta_\Omega = 1$ outside of the thin ellipse and $\beta_f = 1/500$ inside the ellipse.

research in this area. Some related work can be found in [15]. The explicit-jump immersed interface method (EJIIM) [12–14, 16] has been developed for the crack problem with good numerical results. The EJIIM is an extension of the immersed interface method [1,3,8,9]. In the EJIIM approach, the solution and its up to second order derivatives are set as unknowns and coupled with the jump conditions. The discretization leads to a large system of equations and often is solved by iterative method. The study of the stability of the EJIIM is difficult. There is also a large collection of literature for crack problems using finite element formulation.

In this paper we propose a finite difference scheme using simple Cartesian grids to solve the crack problem in both one and two space dimensions. We first use the asymptotic relations to approximate the problem to a two-phase problem with a line interface. Then we use the augmented immersed interface method [2,4,7,10,11] to discretize the problem.

The remaining of paper is organized as follows. In the next section, we discuss the one-dimensional problem. It is easier to understand our method through one-dimensional discussion even though it may not have practical value. In section 3, we present the discussion for two dimensional problems and the numerical method. Finally, some conclusions and acknowledgments are given in the last section.

2. The one dimensional algorithm and analysis

We start with the one dimensional algorithm and analysis by considering the equation

$$(\beta u_x)_x = f(x), \qquad x \in (a,b)$$

where β is a piecewise constant with a finite jump across the crack. We illustrate the problem in Figure 2. Within the domain (a,b), there is a crack that centered at α with the width ϵ. Thus, we assume that $\beta = \beta^o$ in $(\alpha-\epsilon,\, \alpha+\epsilon)$ is different from the β in $(a,\, \alpha-\epsilon)$ and $(\alpha+\epsilon,\, b)$. At the interface $\alpha-\epsilon$ and $\alpha+\epsilon$, the natural jump conditions

$$[u] = 0, \qquad [\beta u_x] = 0, \tag{2}$$

where the jump, for example, $[u]$ is defined as

$$[u]_y = \lim_{x\to y+} u(x) - \lim_{x\to y-} u(x), \tag{3}$$