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TWO-LEVEL METHODS BASED ON THREE CORRECTIONS FOR THE 2D/3D STEADY NAVIER-STOKES EQUATIONS

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Abstract. Two-level finite element methods are applied to solve numerically the 2D/3D steady Navier-Stokes equations if a strong uniqueness condition $(\frac{\|f\|_{-1}}{\|f\|_0})^{\frac{1}{2}} \leq \delta = 1 - \frac{N\|f\|_{-1}}{\nu^2}$ holds, where N is defined in (2.4)-(2.6). Moreover, one-level finite element method is applied to solve numerically the 2D/3D steady Navier-Stokes equations if a weak uniqueness condition $0 < \delta < (\frac{\|f\|_{-1}}{\|f\|_0})^{\frac{1}{2}}$ holds. The two-level algorithms are motivated by solving a nonlinear problem on a coarse grid with mesh size H and computing the Stokes, Oseen and Newton correction on a fine grid with mesh size $h << H$. The uniform stability and convergence of these methods with respect to δ and grid sizes h and H are provided. Finally, some numerical tests are made to demonstrate the effectiveness of one-level method and the three two-level methods.

Key words. Navier-Stokes equations, finite element method, Stokes correction, Oseen correction, Newton correction, two-level method.

1. Introduction

In this report we consider the steady incompressible Navier-Stokes equations:

$$-\nu\Delta u + (u \cdot \nabla)u + \nabla p = f \text{ in } \Omega, \tag{1.1}$$

$$\text{div} u = 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega, \ \int_\Omega p dx = 0, \tag{1.2}$$

which describes a steady flow of the incompressible viscous Newtonian fluid in a bounded domain. Here Ω is a bounded domain in $R^d (d = 2, 3)$ assumed to have a Lipschitz-continuous boundary $\partial\Omega$, $u : \Omega \to R^d$ and $p : \Omega \to R$ are the velocity and pressure, $\nu > 0$ is the viscosity and f represents the given body forces.

Recently, two-level strategy has been studied for steady semi-linear elliptic equations and nonlinear PDEs by Xu [36, 37], and two-level strategy or multi-level strategy has been studied for the steady Navier-Stokes equations by Layton [23], Layton & Tobiska [28], Layton & Lenferink [25, 26] and Layton, Lee & Peterson [27] and Girault and Lions [7] and He et al [14, 17, 18] and Liu and Hou [29], and two level discretizations of flows of electrically conducting, incompressible fluids has been provided by Ervin, Layton and Maubach in [6] . Moreover, a combination of two-level methods and iterative methods for solving the 2D/3D steady Navier-Stokes equations is provided by He et al [20, 21]. As for the nonstationary Navier-Stokes equations, the two-level finite element semi-discretization scheme has been studied by Girault and Lions [9], and the full discretization of the two-level finite element method in space variable x and the one-level backward Euler scheme

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in time variable t have been discussed by Olshanskii [34] and the full discretization of the two-level finite element method in the space-time variables x and t has been studied by He [10, 11] and He et al. [12] Liu and Hou [30, 31] and Hou and Mei[32]. Recently, some multi-level strategy has been studied for the nonstationary Navier-Stokes equations by He et al. [13, 15, 16].

In this paper, for a larger δ satisfying the strong uniqueness condition $\delta \geq (\frac{\|f\|_{-1}}{\|f\|_0})^{\frac{1}{2}}$, we consider three two-level finite element methods by solving a nonlinear Navier-Stokes problem on a coarse grid with mesh size H and computing the Stokes, Oseen and Newton correction on a fine grid with mesh size $h << H$. Moreover, one-level finite element algorithm is applied in the case of the weak uniqueness condition $0 < \delta < (\frac{\|f\|_{-1}}{\|f\|_0})^{\frac{1}{2}}$, where $\frac{\|f\|_{-1}}{\|f\|_0}$ is small for a given f. From some stability and convergence analysis with respect to δ of the one-level finite element method, h and H should be of order $O(\delta)$. And from some stability and convergence analysis with respect to δ of the two-level finite element methods, H should be of order $O(\delta^2)$ and h should be of order $O(H^{\frac{3}{2}})$ or $O(\delta^3)$ in the case of the Stokes and Oseen correction and H should be of order $O(\delta^{\frac{3}{2}})$ and h should be of $O(H^{\frac{3}{2}})$ or $O(\delta^{\frac{9}{4}})$. These facts show that h and H should be very small for small δ. Hence, for the finite element approximation of the 2D/3D steady Navier-Stokes equations, it is better to use one-level finite element method in the case of the weak uniqueness condition and the two-level finite element methods in the case of the strong uniqueness condition.

Remark. It follows from the definition that $\nu = \sqrt{(1-\delta)^{-1}N^{-1}\|f\|_{-1}^{-1}}$. Hence, small δ means small ν. For one-level finite element approximation of the 2D/3D steady Navier-Stokes equations, the Stokes, Oseen and Newton iterative methods can be used, the reader can refer to papers [5, 19, 20, 21].

This paper is organized as follows. In §2 an abstract functional setting of the Navier-Stokes problem is given together with some basic assumption **A0** on Ω for the steady Navier-Stokes problem. In §3 some assumptions **A1-A3** concerning the finite element spaces X_μ and M_μ with $\mu = h, H$ are given, and some uniform stability and convergence with respect to δ of the finite element solution (u_μ, p_μ) are recalled. In §4 the uniform stability and convergence with respect to δ of the two-level finite element method based on the Stokes correction on fine grid is given. In §5 the uniform stability and convergence with respect to δ of the two-level finite element method based on the Oseen correction on fine grid is provided. In §6 the uniform stability and convergence of the two-level finite element method based on the Newton correction are proved. In §7, some numerical tests are made to demonstrate the effectiveness of one-level method and the three two-level methods. In §8 some conclusions are made.

2. Functional Setting of the Navier-Stokes Equations

Let Ω be a convex polygonal/polyhedral domain in R^d. As in [8, 24], we introduce the following Sobolev spaces,

$$X = H_0^1(\Omega)^d,\ Y = L^2(\Omega)^d,\ M = L_0^2(\Omega) = \{q \in L^2(\Omega)\,;\ \int_\Omega q(x)dx = 0\}\,.$$

We denote by $(\cdot,\cdot)$, $\|\cdot\|_0$ the inner product and norm on $L^2(\Omega)$ or $L^2(\Omega)^d$. The space X is equipped with the usual scalar product $(\nabla u, \nabla v)$ and norm $\|\nabla u\|_0$. The subspaces of X and Y are well suited to the incompressible Navier-Stokes equations:

$$V = \{v \in X; \mathrm{div} v = 0 \text{ in } \Omega\},\ V_0 = \{v \in Y; \mathrm{div} v = 0 \text{ and } v \cdot n|_{\partial\Omega} = 0\}.$$