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THE OBSTACLE PROBLEM FOR SHALLOW SHELLS: CURVILINEAR APPROACH

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Abstract. We start with a three-dimensional equilibrium problem involving a linearly elastic solid at small strains subjected to unilateral contact conditions. The reference configuration of the solid is assumed to be a thin shallow shell with a uniform thickness. We focus on the limit when the thickness tends to zero, i.e. when the three-dimensional domain tends to a two-dimensional one. In the generic case, this means that the initial Signorini problem, where the contact conditons hold on the boundary, tends to an obstacle problem, where the contact conditions hold in the domain. When the problem is stated in terms of curvilinear coordinates, the unilateral contact conditions involve a non penetrability inequality which couples the three covariant components of the displacement. We show that nevertheless we can uncouple these components and the contact conditions involve only the transverse covariant component of the displacement at the limit.

Key words. asymptotic analysis, differential geometry, obstacle problem, shallow shells, Signorini conditions.

1. Introduction

The aim of this study was to develop an asymptotic model for a shallow elastic shell which can come into contact with an obstacle. Let us first comment on the so-called obstacle problem. When dealing with the mechanics of a single particle which moves in the presence of some wall, one simply has to ensure that the position of the particle stays on the same side of the wall. But in the case of the mechanics of continuous media, there are two main kinds of problems. First there is the case of a three-dimensional body resting on some support. In this case the contact between the body and the support obviously occurs on a part of the boundary of the body. The corresponding contact conditions have been formulated mathematically by A. Signorini [19] and the equilibrium problem for the body is now classically referred to as a *Signorini problem.* But there exists another case, which seems to be specific to the mechanics of structures, and which can be illustrated by the following example: assuming that a flat membrane clamped at the boundary is pushed up to a wall, then the contact between the membrane and the wall will occur in the membrane, i.e. no longer at the boundary but strictly inside the domain. The corresponding equilibrium problem is the so-called *obstacle problem.*
The present study deals with the justification of the obstacle problem in the case of a shallow shell. The contact conditions will be closely described throughout this study, but we first observe that, due to the existence of a constraint imposed on the position, since the shell cannot cross the obstacle, the contact conditions induce a strong nonlinearity. From the mathematical point of view, this nonlinearity results in a set of constraints which involve inequalities in the displacements and the stresses, and the functional framework is therefore no longer a vector space.
The equilibrium problem has been studied in the case of a plate in [17], where a friction model was added to the description of contact. In the case of a shallow shell involving contact without friction, an asymptotic model has been given and

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justified in the Cartesian framework in [11].
Here we continue to work on these lines by studying this mechanical problem in a system of curvilinear coordinates. Giving and justifying an asymptotic model for a shallow shell in a system of curvilinear coordinates is an interesting result in-itself. In addition the differential geometry framework seems to be more suitable for dealing with the case of general shells. As a matter of fact the Cartesian framework would involve in general several changes of chart, which would make the asymptotic procedure rather complex. It is worth noting that both in the Cartesian and the curvilinear framework, the contact occurs on the boundary in the case of any three-dimensional domain, but the contact will generically occur in the domain in the case of a two-dimensional structural model. In other words the aim here was to prove that a three-dimensional Signorini problem in a domain having a small thickness, namely 2ε in the following, tends to a two-dimensional obstacle problem as ε tends to zero.
This paper is organized as follows.
In section 2 we study a Signorini problem arising in the case of any three-dimensional linearly elastic solid. This is done first in a Cartesian framework and then in a system of curvilinear coordinates. The basic concepts involved are then introduced, in particular the contact conditions, and an existence and uniqueness result for the Signorini problem is recalled, of which the main steps in the proof are outlined.
Section 3 still deals with a three-dimensional domain, which is now the reference configuration of a shallow shell. The contact conditions can be formulated more precisely than in section 2, since the three-dimensional body is now a thin shell. Special attention is paid here to the maps used to build the middle surface and the reference configuration of the shell. The main qualitative situation which results from using curvilinear coordinates is that the contact conditions involve a coupling between all the covariant components of the displacement, whereas only the component normal to the obstacle was involved in the Cartesian case.
In section 4 the three-dimensional domain with a thickness 2ε is changed into a domain with a thickness 2 using a rescaling procedure, and all the data, the unknowns, and the functional framework are rescaled. This makes it possible to perform an asymptotic analysis, which is done in section 5, and to give the limit problem in section 6, where we also give the strong formulation and return to the physical domain in order to have a proper interpretation of the result. The main steps in these sections are classical steps used in most asymptotic analyses; the main difference with previous studies is that the nonlinearity due to the unilateral contact is taken into account here.
The main technical points of interest are given in the Appendix.

2. Formulation of the contact problem in a system of curvilinear coordinates

2.1. The three-dimensional problem in Cartesian coordinates. We first recall the classical contact problem of continuum mechanics. Let $\widehat{\Omega}$ be a domain in $\mathbb{R}^3$, with a system of Cartesian coordinates $\widehat{x} = (\widehat{x}_i)$[1] the closure of which gives the reference configuration of a three-dimensional solid made of an elastic material. When submitted to body forces $\widehat{\boldsymbol{f}} = (\widehat{f}^i) : \widehat{\Omega} \longrightarrow \mathbb{R}^3$, this solid undergoes an elastic displacement field $\widehat{\boldsymbol{u}} = (\widehat{u}_i) : \overline{\widehat{\Omega}} \longrightarrow \mathbb{R}^3$ which solves the set of equilibrium

[1] Latin exponents and indices take their values in the set $\{1;\ 2;\ 3\}$, Greek exponents and indices (except ε) take their values in the set $\{1;\ 2\}$, Einstein's convention for repeated exponents and indices is used and bold letters stand for vectors or vector spaces.