INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 9, Number 3, Pages 543–561

ANALYSIS OF A CARTESIAN PML APPROXIMATION TO THE THREE DIMENSIONAL ELECTROMAGNETIC WAVE SCATTERING PROBLEM

JAMES H. BRAMBLE AND JOSEPH E. PASCIAK

Abstract. We consider the application of a perfectly matched layer (PML) technique applied in Cartesian geometry to approximate solutions of the electromagnetic wave (Maxwell) scattering problem in the frequency domain. The PML is viewed as a complex coordinate shift ("stretching") and leads to a variable complex coefficient equation for the electric field posed on an infinite domain, the complement of a bounded scatterer. The use of Cartesian geometry leads to a PML operator with simple coefficients, although, still complex symmetric (non-Hermitian). The PML reformulation results in a problem which preserves the original solution inside the PML layer while decaying exponentially outside. The rapid decay of the PML solution suggests truncation to a bounded domain with a convenient outer boundary condition and subsequent finite element approximation (for the truncated problem).

For suitably defined Cartesian PML layers, we prove existence and uniqueness of the solutions to the infinite domain and truncated domain PML equations provided that the truncated domain is sufficiently large. We show that the PML reformulation preserves the solution in the layer while decaying exponentially outside of the layer. Our approach is to develop variational stability for the infinite domain electromagnetic wave scattering PML problem from that for the acoustic wave (Helmholtz) scattering PML problem given in [12]. The stability and exponential convergence of the truncated PML problem is then proved using the decay properties of solutions of the infinite domain problem. Although, we do not provide a complete analysis of the resulting finite element approximation, we believe that such an analysis should be possible using the techniques in [6].

Key words. electromagnetic wave scattering problem, Maxwell scattering, Helmholtz equation, PML layer

1. Introduction

In this paper, we consider the application of PML techniques for approximating the solutions of frequency domain Maxwell scattering problems. These problems are posed on unbounded domains with a far field boundary condition given by the Silver-Müller condition. The PML technique which we shall study is one based on Cartesian geometry where each variable is transformed independently.

In an earlier paper Bérenger [2] introduced a PML method for Maxwell's equations in the time domain. This approach was based on constructing a ficticious absorbing layer designed so that plane waves passed into the layer without reflection. The technique involved the introduction of additional variables and equations in the "ficticious material" region. For more analysis on PML applied to time domain problems see [1, 3, 11] and the included references. PML type techniques were also developed in terms of a formal complex change of variable (or stretching) [10, 18]. This approach was especially well suited for frequency domain problems and led to simpler PML formulations more amendable to analysis. Perhaps the simplest and most widely used of the PML variants for frequency domain problems is one which involves a complex change of the Cartesian coordinates.

Received by the editors May, 16, 2011.
1991 *Mathematics Subject Classification.* 65F10, 78M10, 65N30.

A well designed PML reformulation for a scattering problem has the following properties. First, the PML reformulation and the original problem should have the same solution in the "region of interest", i.e., near the scatterer. Second, the solution of the PML reformulation should decay rapidly (usually exponentially) so that it is feasible to truncate the problem to a bounded computational domain with a convenient outer boundary condition. Third, the variational problem on the truncated domain should be stable and thus amendable to finite element approximation. In this paper, we shall show that the Cartesian PML reformulation of the Maxwell scattering problem satisfies all of these properties.

There has been recent work on the stability of PML equations. For polar or spherical PML, [7, 5] have showed stability of the truncated PML approximations to acoustic, electromagnetic and elastic wave scattering problems for stretching functions $\sigma(r) \in C^2(0, \infty)$ which were constant for $r \geq r_1$ (provided that the size of the computational domain is sufficient large). A key ingredient in these analyses is that the coefficients become constant outside of ball of radius r_1 and hence one can apply compact perturbation techniques. Using the particular form of the acoustic scattering two dimensional polar PML equations, Chen and Liu [8] were able to show stability for a stretching of the form

$$\sigma(r) = \sigma_0 \left(\frac{r - r_0}{\rho - r_0} \right)^m$$

for sufficiently large σ_0. The question of stability of the PML equations in the Cartesian case is a much more invoved matter as compact perturbation arguments do not apply. Recently, Cartesian PML approximations to acoustic scattering problems were successfully analyzed in [12] and [4] for PML functions $\sigma(x)$ which are constant for large x. The first paper, [12], shows that the truncated PML equations are stable if the computational domain is sufficiently large. The second [4] is more general and further proves stability provided that the product of the domain size and σ_0 is sufficiently large. We also mention that stability through the PML layer was provided for two dimensional acoustic PML problems with piecewise constant σ by Chen and Zheng [9].

The goal of this paper is provide stability estimates for the truncated Cartesian PML formulations of the Maxwell scattering problem. To do this, we shall show stability on a sequence of domains, one leading to the next. Specifically, we denote the domain of the (bounded) scatterer by $\Omega \subset \mathbb{R}^3$ and the interior of its complement by Ω^c. We show that:

- PML Cartesian stability for the acoustic probem implies similar stability for the Maxwell problem on all of $\mathbb{R}^3$.
- PML Cartesian stability for the Maxwell problem on $\mathbb{R}^3$ implies similar stability on Ω^c.
- PML Cartesian stability for the Maxwell problem on Ω^c implies similar stability on the computational domain $(-M, M)^3 \setminus \bar{\Omega}$ for sufficiently large M.

The outline of the remainder of the paper is as follows. In Section 2, we formulate the Maxwell scattering problem. The PML operators are defined in Section 3. Sections 4 and 5 prove variational stability of the PML problem in $\mathbb{R}^3$ and Ω^c, respectively. Section 6 shows variational stability on the truncated domain and exponential convergence of the coresponding solution to the solution of the original problem on the region of interest. For simplicity, we only consider the case where