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# SHOOTING METHODS FOR NUMERICAL SOLUTIONS OF EXACT CONTROLLABILITY PROBLEMS CONSTRAINED BY LINEAR AND SEMILINEAR 2-D WAVE EQUATIONS

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*This paper is dedicated to Max Gunzburger on the occasion of his 60th birthday*

**Abstract.** Numerical solutions of exact controllability problems for linear and semilinear 2-d wave equations with distributed controls are studied. Exact controllability problems can be solved by the corresponding optimal control problems. The optimal control problem is reformulated as a system of equations (an optimality system) that consists of an initial value problem for the underlying (linear or semilinear) wave equation and a terminal value problem for the adjoint wave equation. The discretized optimality system is solved by a shooting method. The convergence properties of the numerical shooting method in the context of exact controllability are illustrated through computational experiments.

**Key Words.** Controllability, finite difference method, distributed control, optimal control, parallel computation, shooting method, wave equation.

## 1. Introduction

In this paper, we consider an optimal distributed control approach for solving the exact distributed controllability problem for two-dimensional linear or semilinear wave equations defined on a time interval $(0,T)$, and spatial domain $\Omega$ in $\mathbb{R}^2$. The exact distributed controllability problem we consider is to seek a distributed control $f$ in $L^2((0,T)\times\Omega)$ and a corresponding state $u$ such that the following system of equations hold:

$$
\left\{
\begin{array}{l}
u_{tt} - \Delta u + \Psi(u) = f \quad \text{in } Q \equiv (0,T)\times\Omega\,,\\
u|_{t=0} = w \quad \text{and} \quad u_t|_{t=0} = z \quad \text{in } \Omega\,,\\
u|_{t=T} = W \quad \text{and} \quad u_t|_{t=T} = Z \quad \text{in } \Omega\,,\\
u|_{\partial\Omega} = 0 \quad \text{in } (0,T)\,,
\end{array}
\right.
\tag{1.1}
$$

where $w$ and $z$ are given initial conditions defined on $\Omega$, $W \in L^2(\Omega)$ and $Z \in H^{-1}(\Omega)$ are prescribed terminal conditions, $f$ in $L^2((0,T)\times\Omega)$ is the distributed control, and $\Psi(u)$ is a given function on $\mathbb{R}$.

The exact boundary controllability problems are well known for linear and semilinear cases; see e.g.,[5, 14, 15, 17, 18, 20, 21, 23, 24]. In these problems there are basically two classes of computational methods in the literature. The first class is known Hilbert Uniqueness Method (HUM); see, e.g., [9, 11, 14, 16, 22]. The approximate solutions obtained by the HUM-based methods in general do not seem

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to converge (even in a weak sense) to the exact solutions as the temporal and spatial grid sizes tend to zero. Methods of regularization including Tychonoff regularization and filtering that result in convergent approximations were introduced in those papers on HUM-based methods. The second class of computational methods for boundary controllability of the linear wave equation was those based on the method proposed in [10]. One solves a discrete optimization problem that involves the minimization of the discrete boundary $L^2$ norm subject to the undetermined linear system of equations formed by the discretization of the wave equation and the initial and terminal conditions. This approach was implemented in [8]. The computational results demonstrated the convergence of the discrete solutions when the exact minimum boundary $L^2$ norm solution is smooth. In the generic case of a non-smooth exact minimum boundary $L^2$ norm solution the computational results of [8] exhibited at least a weak $L^2$ convergence of the discrete solutions.

In this paper we develop an alternate numerical method which allows us to apply distributed or boundary control to the exact controllability problems. Ultimately we test the exact boundary controllability problems, but it is beyond the work, and we will present the result in a separate paper. The results in [19] were limited to the one dimensional case. In this paper, we extend those results to the two dimensional case.

We will study numerical methods for optimal control and controllability problems associated with the linear and semilinear wave equations. We are particularly interested in investigating the relevancy and applicability of high performance computing (HPC) for these problems. As a prototype example of optimal control problems for the wave equations we consider the following distributed optimal control problem: choose a control $f$ and a corresponding $u$ such that the pair $(u, f)$ minimizes the cost functional

$$\begin{aligned}\mathcal{J}(u,f) =&\frac{\alpha}{2}\int_0^T\int_\Omega K(u)\,d\mathbf{x}\,dt + \frac{\beta}{2}\int_\Omega \Phi_1(u(T,\mathbf{x}))\,d\mathbf{x} + \frac{\gamma}{2}\int_\Omega \Phi_2(u_t(T,\mathbf{x}))\,d\mathbf{x}\\ &+\frac{1}{2}\int_0^T\int_\Omega |f|^2\,d\mathbf{x}\,dt\end{aligned} \tag{1.2}$$

subject to the wave equation

$$\begin{cases} u_{tt} - \Delta u + \Psi(u) = f \quad \text{in } Q \equiv (0,T)\times\Omega\,,\\ u|_{\partial\Omega} = 0, \quad \text{in } (0,T)\,,\\ u(0,\mathbf{x}) = w(\mathbf{x}) \text{ and } u_t(0,\mathbf{x}) = z(\mathbf{x}) \quad \text{in } \Omega\,. \end{cases} \tag{1.3}$$

Here $\Omega$ is a bounded spatial domain in $\mathbb{R}^d$ ($d = 1$ or 2 or 3) with a boundary $\partial\Omega$; $f$ is a distributed control and $u$ is the corresponding state. Also, $K$, $\Phi$ and $\Psi$ are $C^1$ mappings (for instance, we may choose $K(u) = (u-U)^2$, $\Psi(u) = 0$, $\Psi(u) = u^3 - u$ and $\Psi(u) = e^u$, $\Phi_1(u) = (u(T,\mathbf{x})-W)^2$, $\Phi_2(u) = (u_t(T,\mathbf{x})-Z)^2$, where $U, W, Z$ are target functions). Moreover we assume that initial conditions $w$ and $z$ are smooth enough to be well defined the given problem; see e.g.,[4]. Also we suppose that nonlinearity $\Psi(u)$ does not alter the regularity of the solution in the wave equation.

Of particular interest to us is the case of large $\alpha$, $\beta$ and $\gamma$; our computational experiments of the proposed numerical method will be performed exclusively for this case. Our interest in this case stems from the fact that the optimal control problem can be viewed as an approximation to the exact distributed controllability problem (1.1).

Such control problems are classical ones in the control theory literature; see, e.g., [12] for the linear case and [13] for the nonlinear case regarding the existence of