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STABILIZATION OF NAVIER-STOKES EQUATIONS BY BOUNDARY FEEDBACK

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Abstract. In this paper, we consider the stabilization of steady state solutions to Navier-Stokes equations by boundary feedback control. The feedback control is determined by solving a linear quadratic regulator problem associated with the linearized Navier-Stokes equations. The control is effected through suction and blowing at the boundary. We show that the linear feedback control provides global exponential stabilization of the steady state solutions to the Navier-Stokes equations for arbitrary Reynolds number. This feedback is shown to provide global stability in both L^2 and H^1-norms.

Key Words. Navier-Stokes equation, Riccati equation, stabilization, feedback control, linear quadratic regulator.

1. Introduction

Control of fluid flows for the purpose of achieving some desired objective is crucial to many technological and scientific applications. The invention fast micro devises such as MEMS to actually implement these controls has increased the interest in this area. Control design for fluid dynamical systems is hindered by the intrinsic difficulties caused by the nonlinearity and infinite dimensionality of the Navier-Stokes equations that govern fluid flows. In the recent past, great advances have been made in theoretical and computational analysis of optimal control of fluids, see for e.g. [7, 21, 8, 9, 6, 12, 17, 11, 3, 16].

In this article, we address the stabilization problem for viscous flows modeled by the Navier-Stokes equations which has applications in turbulence and drag reduction. It is well-known that the steady state solutions to Navier-Stokes equations might be unstable for high Reynolds number. Our objective here is to develop a boundary feedback control to stabilize the steady solutions of Navier-Stokes equations in bounded domain. The control is effected through suction and blowing on the boundary and we do not make any distinction in our analysis here as to wall normal blowing and suction [16, 17] or tangential velocity actuation [18] as we did in those computational analysis works. We wish to find a boundary control in feedback form on a part of the boundary so that the corresponding system with this control substituted is globally exponentially stable for arbitrary Reynolds number. Motivated by the Lyapunov stability theory for finite dimensional nonlinear ordinary differential equations, we propose a linear feedback control using the algebraic Riccati equation associated with an infinite time horizon linear quadratic regulator (LQR) problem.

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In order to state our problem more precisely, consider the abstract evolution problem

$$\frac{d\mathbf{y}}{dt} = F(\mathbf{y}, g)\,, \quad \mathbf{y}(0) = \mathbf{y}_0\,, \tag{i}$$

where $g(t)$ is the control and $F : \mathbf{X} \times U \to V$ is a nonlinear mapping. The corresponding steady state problem is

$$F(\bar{\mathbf{y}}, \bar{g}) = 0\,. \tag{ii}$$

Suppose $(\bar{\mathbf{y}}, \bar{g}) \in \mathbf{X} \times U$ is a given steady state solution of (ii). The problem of stabilizing the unsteady solution $\mathbf{y}$ of (i) near the steady state solution $\bar{\mathbf{y}}$ with a prescribed rate $\sigma > 0$ is to find the control $g(t)$ such that the solution $\mathbf{y}(t)$ of the unsteady problem with this $g(t)$ satisfies

$$\|\mathbf{y}(t) - \bar{\mathbf{y}}\| \leq ce^{-\sigma t}\,, \quad t \in (0, \infty)\,. \tag{iii}$$

The control $g(t)$ is called feedback if there is an operator $K : \mathbf{X} \to U$ such that $g(t) = K(\mathbf{y})$. The feedback stabilization problem that we consider here can be formulated, for the above abstract evolution equation (i), as follows:
Given a steady-state solution $(\bar{\mathbf{y}}, \bar{g})$ *of (ii), find an operator* $K : \mathbf{X} \to U$ *such that the solution* $\mathbf{y}(t)$ *of the problem*

$$\frac{d\mathbf{y}}{dt} = F(\mathbf{y}, K(\mathbf{y}))\,, \quad \mathbf{y}(0) = \mathbf{y}_0 \tag{iv}$$

satisfies (iii).
Our objectives are to first derive a feedback control using the theory of optimal linear quadratic regulator over an infinite time horizon for the linearized Navier-Stokes equations when the control is on the boundary and to show that the resulting linear feedback control globally stabilizes the nonlinear closed-loop problem in the sense stated above for arbitrary Reynolds number. In particular we will derive stability estimates in both L^2 and H^1-norms.

Other related works that use optimal feedback control theory to flow stabilization can be found in [3, 5]. In [3] robust feedback control is used for stabilization of Navier-Stokes equations by distributed control on the whole domain. In [5], stabilization by boundary control of two dimensional Euler equations for incompressible flow is considered.

The paper is organized as follows. In the rest of this section, we present the notations that we will use and the mathematical preliminaries we will need to present our results in the sequel. In Section 2, we formulate our stabilization problem. In Section 3, we present the feedback control design and study the stability of the nonlinear closed-loop system. The stability analysis is carried out with the help of Lyapunov techniques and Galerkin methods. In Section 4, we conclude the paper.

1.1. Notation and Preliminaries. We introduce the following standard notations over a bounded, connected, open set Ω in $\mathbb{R}^2$ with boundary $\Gamma \in C^2$. Let $\mathbf{n}$ denote the unit normal vector to Γ. For $p \in [1, \infty)$, let $L^p(\Omega)$ denote the measurable real-valued functions v on Ω for which $\int_\Omega |v(\mathbf{x})|^p d\mathbf{x} < \infty$. In addition, let $L^\infty(\Omega)$ denote the measurable real-valued functions that are bounded, or at least essentially bounded. For $v \in L^p(\Omega)$, we may define

$$\|v\|_p \equiv \left(\int_\Omega |v(\mathbf{x})|^p d\mathbf{x}\right)^{\frac{1}{p}}\,, \quad \text{for } 1 \leq p < \infty\,,$$

$$\|v\|_\infty \equiv \operatorname{ess\,sup}_{\mathbf{x} \in \Omega} |v(\mathbf{x})|\,.$$