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A REVIEW OF THEORETICAL MEASURE APPROACHES IN OPTIMAL SHAPE PROBLEMS

ALIREZA FAKHARZADEH JAHROMI AND HAJAR ALIMORAD

Abstract. Some optimal shape design problems lack classical solutions, or at least, the existence of such solutions is far from being straightforward. In such cases, to obtain an optimal solution, a variety of methods have been employed. In this study, we review the works that used measures which can basically be divided in two groups: using Young measures and embedding process (Shape-measure method). We also survey the advantages and disadvantages of these two methods and investigate their improved version in the presented works and applications.

Key words. Young measure, radon measure, atomic measure, optimal shape, shape-measure, linear programming problem, relaxed problem.

1. Introduction

The field of shape optimization problems has recently attracted the attention of many scholars. These researchers argue for a number of applications in physics and engineering which require a focus on shapes rather than on parameters or functions. The purpose of such applications is to modify admissible shapes so that they can comply with a given cost function which needs to be optimized. In general, the study of optimal shape design (OSD) tries to answer the question of "*What is the best shape for a physical system*". The term OSD is used whenever a function has to be minimized with respect to a particular geometric element (or elements) like curve, domain, or point.

It is well-known that a measurable set, like a shape, can be considered as a measure. On the other hand, a question which comes into mind is when a measure can be considered as a shape. This is the base for special techniques in optimal shape (OS) and OSD problems which try to determine the optimal shape as a measure. The main aim of this paper is to present a complete survey on the OS and OSD theoretical measure based method. As literature show, two measures are used in this application: Young measure and positive Radon measure (we also even see traces of atomic measure in these works and it is necessary to remind that the atomic measure is a part of the methods which are based on these two measures as will be explained later).

We can use Young measures mainly as a tool to organize and comprehend the behavior of sequences of functions with respect to integral functionals; in particular, relaxed optimization problems whose generalized solutions come from a sequence of functions generate the generalized minimizer of the original problem. Young measures can be used to describe these relaxed formulations of different types of optimization problems [68]. There is an extensive literature about the application of Young measures in different subjects; for instance control theory [9], differential equations ([78] and [11]). A key feature of these kinds of measures is their capacity to capture the oscillations of minimizing sequences of non-convex variational problems, and many applications appear in ([10] and [35]); for example in models of

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elastic crystals. Some other properties of Young measures can be found in many studies like [45] and [46].

Using Radon measures for solving optimal control problems based on the idea of L. C. Young (see [81]) was applied for the first time in [80] and the method was theoretically established by Rubio [75]. This approach is based on the definition of a measure on a product space, of which control is just a factor. In this way, one constructs a linear optimization problem corresponding to the control problem. This method was extended and improved by others such as Kamyad et al. [40], [42] and [43] and Farahi et al. [25], [6] and [26] to name some. In 1996, Fakharzadeh [11] used this method to solve some optimal shape design problems governed by elliptic equations in two dimensions. The similar idea for solving OSD problems in a relaxed sense is named Shape-measure. The measures used in Shape-measure method are uniquely defined while those used in Young measure are not defined uniquely [56].

It is necessary to indicate that measures like Occupation measure applied to solve optimal control problems [47] have not yet been used for solving OS and OSD problems. Therefore, this paper is divided into two main sections. In the first one, the OS and OSD problems which are solved by using Young measures are reviewed. In the next section, the application of Radon measures (Shape-measure method) for solving the mentioned problems are surveyed. In each section, to make the readers more familiar with the studies, one of those studies has been explained relatively extensively. Then, an attempt has been made to explain all the studies conducted on the basis of these methods with a view of expanding the method (in terms of the model of the problem, method of utilizing the measures and the way of transferring the main problem into the measures space). It is necessary to remind that since generally used methods for shape optimization have a general framework, applying them for each problem depends on the kind of the problem and the creativity of the person who uses them. In this regard, in each application of these two methods we are faced with some novelties which have to be represented in this review carefully. Finally, the critical and comparative analysis of the two methods for solving OS and OSD problems are presented. Since this paper may not be that comprehensive, we hereby apologize to those authors and readers whose works could not be cited in this study.

2. Young measure based methods

Young measures which were originally conceived as 'generalized curves' by L. C. Young complete sets of ordinary curves in the calculus of variations. It has been proved that a broad class of problems in the calculus of variations has solutions in the form of these generalized curves [2]. Consider an energy functional which lacks the property of lower semi-continuity, in such circumstances, the infimum of energy is achieved only in some generalized sense while a minimizing sequence may develop finer and finer oscillations, reminiscent of a finely twinned microstructure [44]. Young measures are used in optimization problems (and shape optimization) where a local, integral cost functional is to be minimized in a suitable class of functions which often lack optimal solutions because of the presence of some non-convexity. In such cases, a single function is unable to reproduce the optimal behavior due precisely to this lack of optimal solutions, and one must resort to sequences (the so-called minimizing sequences) in order to comprehend the main features of optimality.