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A SUBGRID VISCOSITY LAGRANGE-GALERKIN METHOD FOR CONVECTION-DIFFUSION PROBLEMS

RODOLFO BERMEJO, PEDRO GALÁN DEL SASTRE, AND LAURA SAAVEDRA

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This paper is dedicated to Francisco Lisbona on occasion of his 65th birthday

Abstract. We present and analyze a subgrid viscosity Lagrange-Galerkin method that combines the subgrid eddy viscosity method proposed in W. Layton, A connection between subgrid scale eddy viscosity and mixed methods. Appl. Math. Comp., 133: 147-157, 2002, and a conventional Lagrange-Galerkin method in the framework of $P_1\oplus$ cubic bubble finite elements. This results in an efficient and easy to implement stabilized method for convection dominated convection-diffusion-reaction problems. Numerical experiments support the numerical analysis results and show that the new method is more accurate than the conventional Lagrange-Galerkin one.

Key words. Subgrid viscosity, Lagrange-Galerkin, finite elements, convection-diffusion-reaction problems.

1. Introduction

The design of efficient and accurate convection-diffusion algorithms is of significant importance in the computational fluid dynamics community, in particular, when the transport terms of the equations describing the mathematical model become dominant with respect to the diffusion ones. In this case there appear a large variety of spatial-temporal scales that have to be properly resolved in order to obtain a numerical solution sufficiently close to the exact one. The prototype problem to test a convection-diffusion algorithm considers a passive substance, the concentration of which is denoted by $c\,(x,t)$, in a bounded domain $D \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with Lipschitz continuous boundary ∂D, such that

$$(1)\qquad \begin{cases} \dfrac{\partial c}{\partial t} + \mathbf{b}\cdot\nabla c - \varepsilon\Delta c + \alpha c = f \ \text{ in } D \quad \times (0,T)\,, \\ \\ c = 0 \ \text{ on } \ \partial D \ \times [0,T] \text{ and } \ c(x,0) = v \ \text{ in } \ D, \end{cases}$$

where $\mathbf{b}$ is the velocity vector that for simplicity we shall assume that vanishes on $\partial D\times[0,T]$, $\varepsilon > 0$ is the diffusion coefficient, and $[0,T]$ denotes the time interval. We assume that $\mathbf{b} \in L^\infty(0,T;W^{1,\infty}(D)^d)$, $f \in L^2(0,T;L^2(D))$, $\alpha \in C([0,T];C(D))$, $v \in H^1(D)$, and $\varepsilon \ll \|\mathbf{b}\|_{L^\infty(D\times(0,T))^d}$; moreover, there exists a positive constant $\underline{\alpha}$ such that for all $(x,t) \in D \times [0,t])$, $\alpha(x,t) \geq \underline{\alpha} \geq 0$. In many places the material derivative $\frac{Dc}{Dt} := \frac{\partial c}{\partial t} + \mathbf{b}\cdot\nabla c$ is used.

The dimensionless form of this equation contains the so-called Péclet number Pe defined as $Pe = \frac{UL}{\varepsilon}$, where U and L represent a characteristic velocity and a characteristic length scale respectively. The numerical treatment of this problem is difficult when Pe is large enough because the diffusion term, $\varepsilon\Delta c$, may be considered as a perturbation to the convective term, $\frac{\partial c}{\partial t} + \mathbf{b}\cdot\nabla c$, in regions where $c\,(x,t)$ is smooth so that in these regions the dynamics of the solution is mainly governed

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by $\frac{\partial c}{\partial t} + \mathbf{b} \cdot \nabla c$, the latter mathematical expression represents the change of c along the characteristic curves (or trajectories of the flow particles) of the hyperbolic operator $\frac{\partial}{\partial t} + \mathbf{b} \cdot \nabla$. But the existence of boundary conditions to be satisfied by $c\,(x,t)$ on $\partial D \times (0,T)$ is incompatible with the hyperbolic character of $\frac{\partial c}{\partial t} + \mathbf{b} \cdot \nabla c$; hence, the imposition of the boundary conditions will lead to the appearance of a region near the boundary where the solution has to accommodate to satisfy the boundary conditions. This region is termed boundary layer, and one can show through perturbation analysis that its width is $O(Pe^{-\alpha})$, $0 < \alpha < 1$. Therefore, for high Péclet numbers the boundary layer is narrow and, consequently, the solution will develop a strong gradient in it. It is well known, see for instance [21], that numerical methods based on Galerkin projection (either finite elements, or spectral methods, or hp finite elements) have serious drawbacks in solving the convection-diffusion equation at high Pe numbers for the following reasons: (1) they will develop spurious oscillations (Gibbs phenomenon), which pollute the numerical solution, unless the boundary layers are properly resolved; this means that one has to allocate many mesh-points in regions close to the boundary layers to suppress the spurious oscillations; (2) the error of standard Galerkin methods is of the form

$$\max_{t_n} \|c(t_n) - c_h^n\|_{L^2(D)} = C_G \left(h^m + \Delta t^q\right),$$

where h is the mesh size, Δt the size of the time step, m and q positive real numbers, and the constant C_G is of the form

$$C_G \sim Pe \exp(t_n \max_{D \times [0,t_n]} |\mathbf{b}|\, Pe).$$

Issue (1) and numerical stability reasons require the use of implicit time stepping schemes to advance in time the numerical solution and, consequently, the use of non-symmetric solvers; the latter being less efficient than solvers for symmetric systems.

Following different approaches, such as Eulerian, Eulerian-Lagrangian, and Lagrangian, several algorithms have been devised in the framework of Galerkin methods to overcome the drawbacks described above. In the Eulerian approach one calculates mesh-point values of c at time instants t_n, formulating the numerical method in a fixed mesh with the purpose of suppressing the wiggles without damaging the accuracy of the method. To this respect, we shall refer to the SUPG (Stream-Upwind-Petrov-Galerkin) and the Galerkin/least squares algorithms developed by Hughes and coworkers [6], [17] for convection-diffusion problems of a passive substance, as well as for the Navier-Stokes equations and conservation laws; the edge stabilization methods [7]; the subgrid viscosity methods of [11] and [20], and finally the variational multiscale methods introduced by [16] and further developed by many people.

In the Lagrangian approach one attempts to devise a stable numerical method by allowing the mesh-points to follow the trajectories of the flow. The problem now is that the mesh undergoes large deformations after a number of time steps, due to stretching and shearing, consequently some sort of remeshing has to be done in order to proceed with the calculations. The latter may become a source of large errors.

In the Eulerian-Lagrangian approach the purpose is to get a method that combines the good properties of both the Eulerian and Lagrangian approaches. There have been various methods trying to do so, among them we shall cite the characteristics streamline diffusion (CSD) method, the Eulerian-Lagrangian localized adjoint