Convolution Quadrature Methods for Time-Space Fractional Nonlinear Diffusion-Wave Equations

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Abstract. Two second-order convolution quadrature methods for fractional nonlinear diffusion-wave equations with Caputo derivative in time and Riesz derivative in space are constructed. To improve the numerical stability, the fractional diffusion-wave equations are firstly transformed into equivalent partial integro-differential equations. Then, a second-order convolution quadrature is applied to approximate the Riemann-Liouville integral. This deduced convolution quadrature method can handle solutions with low regularity in time. In addition, another second-order convolution quadrature method based on a new second-order approximation for discretising the Riemann-Liouville integral at time $t_k-1/2$ is constructed. This method reduces computational complexity if Crank-Nicolson technique is used. The stability and convergence of the methods are rigorously proved. Numerical experiments support the theoretical results.

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1. Introduction

Fractional partial differential equations naturally arise in anomalous diffusion with random walk processes due to non-local properties of fractional integrals and fractional derivatives. The fractional anomalous diffusion models are obtained by replacing the integer-order calculus operators in the classical diffusion equations by fractional operators for...
the memory effects [26, 34, 37, 42]. In particular, time and space fractional diffusion-wave equations can interpolate the diffusion and the wave phenomena and describe processes with spatial non-local dependence. Therefore, such models are widely used for description of viscoelastic damping materials, diffusion images of human brain tissues, etc. [11, 16, 22, 29, 35].

Since it is very difficult or often impossible to obtain analytical solutions of fractional diffusion-wave equations [1, 3, 25, 27, 28], numerical methods are required. There is a vast literature on approximation methods for time or time-space fractional linear diffusion-wave equations [2, 5, 6, 12, 13, 19–21, 30, 33, 36, 38, 41, 43–45]. Using a classical \((3 - \alpha)\)-order approximation for the Caputo derivative, Sun and Wu [38] constructed a finite difference scheme and studied its stability and convergence. Li et al. [19] applied a finite difference method in time and finite element method in space to time-space fractional diffusion-wave equations and investigated semidiscrete and fully discrete numerical approximations. Liu et al. [21] considered numerical methods for multi-term time-fractional wave-diffusion equations. Using equivalent partial integro-differential equations, Huang et al. [13] constructed two finite difference schemes for a class of time fractional diffusion-wave equations and proved their first- and second-order convergence in temporal and spatial directions, respectively. Mustapha and Schötzau [30] established the well-posedness of an \(hp\)-version of time-stepping discontinuous Galerkin method for fractional diffusion-wave evolution problems, derived error estimates in a nonstandard norm and showed exponential convergence in the number of temporal degrees of freedom for solutions with singular behavior near \(t = 0\).

On the other hand, Wang and Vong [41] used a weighted and shifted Grünwald difference operator and compact difference technique to construct a higher order scheme for a time fractional diffusion-wave equation. Bhrawya et al. [2] presented a spectral numerical method for fractional diffusion-wave and fractional wave equations with damping. The method is based on the Jacobi \(\tau\)-spectral procedure and Jacobi operational matrix for fractional Riemann-Liouville integrals. Zeng [44] proposed second-order in time and space stable and conditionally stable finite difference schemes for time fractional super-diffusion equation based on the fractional trapezoidal rule and the generalised Newton-Gregory formula. Ye et al. [43] derived a compact difference scheme for a distributed-order time-fractional diffusion-wave equation, and proved its unique solvability, stability and convergence. Chen and Li [5] used equivalent integro-differential equations and product trapezoidal rule to construct a compact finite difference scheme for fractional diffusion-wave equations. Chen et al. [6] considered a second-order backward differentiation formula alternating direction implicit difference for two-dimensional time fractional diffusion-wave equations.

The problems arising in numerical solution of non-linear fractional diffusion-wave equations are more complex. Nevertheless, Dehghan and Abbaszadeh [7] constructed a finite difference-spectral element method and studied its stability and convergence. This method performs better than other existing methods. Huang and Yang [14] combined the spectral Galerkin method in space and the fractional trapezoidal method in time having the spectral accuracy in space.
Let $1 < \alpha, \beta < 2$ and $\frac{RL}{0} D^\beta_x$ and $\frac{RL}{x} D^\beta_L$ be the Riemann-Liouville fractional derivatives

\[
\frac{RL}{0} D^\beta_x u(x, t) := \frac{1}{\Gamma(2-\beta)} \left( \frac{\partial}{\partial x} \right)^2 \int_0^x (x-s)^{1-\beta} u(s, t) ds,
\]

\[
\frac{RL}{x} D^\beta_L u(x, t) := \frac{1}{\Gamma(2-\beta)} \left( -\frac{\partial}{\partial x} \right)^2 \int_x^0 (s-x)^{1-\beta} u(s, t) ds,
\]

and let $\frac{C}{0} D^\alpha_t u(x, t)$ and $\frac{\partial}{\partial |x|^\beta} u(x, t)$ be, respectively, temporal Caputo and spatial Riesz fractional derivatives — i.e.

\[
\frac{C}{0} D^\alpha_t u(x, t) := \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \frac{\partial^2 u(x, s)}{\partial s^2} ds,
\]

\[
\frac{\partial}{\partial |x|^\beta} u(x, t) := \frac{1}{2\cos(\beta \pi/2)} \left[ \frac{RL}{0} D^\beta_x u(x, t) + \frac{RL}{x} D^\beta_L u(x, t) \right].
\]

We consider the time-space fractional nonlinear diffusion-wave equation

\[
\frac{C}{0} D^\alpha_t u(x, t) = K_c \frac{\partial}{\partial |x|^\beta} u(x, t) + g(u) + f(x, t), \quad 0 < x < L, \quad 0 < t \leq T, \quad (1.1)
\]

\[
u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad 0 < x < L,
\]

\[
u(0, t) = u(L, t) = 0, \quad 0 < t \leq T,
\]

where $K_c$ is a positive constant, $g(u)$ a nonlinear function satisfying the Lipschitz condition and such that $g(0) = 0$, $f(x, t)$ a linear function. If $u(x, 0) = \varphi(x) \neq 0$ and $u_t(x, 0) = \psi(x) \neq 0$, the initial value conditions can be homogenised by the transformation

\[
\tilde{u}(x, t) = u(x, t) - \varphi(x) - t\psi(x).
\]

We note that various important models in quantum mechanics, plasma physics and Bose-Einstein condensation, including fractional Schrödinger equations and fractional Klein-Gordon equations [10, 15, 17, 39] can be considered as special cases of the Eq. (1.1).

In this work we consider two second-order convolution quadrature methods, the main advantage of which is that the second-order approximations in time can be achieved under low regularity assumptions.

The remainder of this paper is organised as follows. Section 2 contains necessary auxiliary information. In Section 3, we introduce a second-order convolution quadrature method and show its convergence and stability. In Section 4, a second-order approximation for Riemann-Liouville integral at time $t_{k-1/2}$ is used to develop another second-order convolution quadrature method. Section 5 presents the results of numerical experiments, and our concluding remarks are in Section 6.
2. Preliminaries

According to [8, 13], the Eq. (1.1) is equivalent to the following partial integro-differential equation

\[ u_t(x, t) = K_c \cdot \partial^{\alpha - 1} u_{|x|} + \partial^\alpha g(u(x, t)) + F(x, t), \tag{2.1} \]

where \( \partial^\alpha \) is the Riemann-Liouville fractional integral operator of order \( \alpha > 0 \),

\[ \partial^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} g(s)ds, \]

and \( F(x, t) = \partial^\alpha f(x, t) \) is explicitly computable.

Let \( \tau := T/N \), \( t_i := n\tau \), and the time domain \((0, T)\) is covered by the mesh \( \Omega_t := \{ t_i | 0 \leq i \leq N \} \). Similarly, we set \( h := L/M \), \( x_i := ih \), \( 0 \leq i \leq M \) and consider the uniform mesh \( \Omega_h = \{ x_i | 0 \leq i \leq M \} \) on the spatial interval \([0, L]\).

Let us introduce a few more notations. If \( \omega = \{ \omega^n | 0 \leq i \leq N \} \) is a grid function on \( \Omega_t \) and \( V = \{ V_i | 0 \leq i \leq M, V_0 = V_M = 0 \} \), \( W = \{ W_i | 0 \leq i \leq M, W_0 = W_M = 0 \} \) are grid functions on \( \Omega_h \), then

\[ \delta_t \omega^{n+1/2} := \frac{\omega^{n+1} - \omega^n}{\tau}, \quad \langle V, W \rangle := \sum_{i=1}^{M-1} h V_i W_i, \quad \|V\|^2 := \langle V, V \rangle. \]

Besides, in what follows, the symbol \( C \) always refers to a generic constant, whose value may be different from one line to another.

**Lemma 2.1.** Let \( 0 < \mu \leq 1 \) and \( \omega_{t}^{(\mu)} \) be the weights generated by the function \((3/2 - 2\gamma + z^2)/2^\gamma\). Then

(a) If \( g(t) \in C^2([0, T]) \) and \( g(0) = g'(0) = 0 \), then

\[ \left| \partial^\mu g(t_{n+1}) - \tau^\mu \sum_{k=0}^{n+1} \omega_{t}^{(\mu)} g(t_k) \right| \leq C \max |g''(t)| \tau^2. \]

(b) If \( g(t) = \varphi(t^\gamma) \) and \( 1 < \gamma < 2 \), then

\[ \left| \partial^\mu g(t_{n+1}) - \tau^\mu \sum_{k=0}^{n+1} \omega_{t}^{(\mu)} g(t_k) \right| \leq C t_{n+1}^{\mu + 2 - \frac{1}{2}} \tau^2. \]

**Proof:** The weight coefficients \( \omega_{t}^{(\mu)} \) are derived in [23] and the inequalities above follows from [24, Theorems 3.1 and 5.2].
The weights $\omega_k^{(\mu)}$ can be obtained immediately without Fourier transform — viz. using the expansion

\[
\left( \frac{(1-z)(3-z)}{2} \right)^{-\mu} = \left( \frac{3}{2} \right)^{-\mu} (1-z)^{-\mu} \left( 1 - \frac{z}{3} \right)^{-\mu} = \left( \frac{3}{2} \right)^{-\mu} \left( \sum_{k=0}^{\infty} g_k^{(\mu)} z^k \right) \left( \sum_{k=0}^{\infty} g_k^{(\mu)} \left( \frac{z}{3} \right)^k \right) = \left( \frac{3}{2} \right)^{-\mu} \sum_{k=0}^{\infty} \sum_{j=0}^{k} g_j^{(\mu)} g_{k-j}^{(\mu)} \frac{z^k}{3^{k-j}},
\]

where $g_0^{(\mu)} = 1$, $g_k^{(\mu)} = (-1)^k \binom{-\mu}{k}$, $k \geq 1$, we find out that

\[
\omega_k^{(\mu)} = \left( \frac{3}{2} \right)^{-\mu} \sum_{j=0}^{k} g_j^{(\mu)} g_{k-j}^{(\mu)} \frac{1}{3^{k-j}}.
\]

**Lemma 2.2.** Let $0 < \mu \leq 1$ and $k \geq 0$. Then

(a) If $g(t) \in C^2([0, T])$ and $g(0) = g'(0) = 0$, then

\[
o_0 J_t^\mu g(t_{k+1/2}) = \frac{1}{2} \left[ o_0 J_t^\mu g(t_{k+1}) + o_0 J_t^\mu g(t_k) \right] + O(\tau^2).
\]

(b) If $g(t) = O(t^\gamma)$ and $1 < \gamma < 2$, then

\[
o_0 J_t^\mu g(t_{k+1/2}) = \frac{1}{2} \left[ o_0 J_t^\mu g(t_{k+1}) + o_0 J_t^\mu g(t_k) \right] + O(t^{\mu+\gamma-2} \tau^2).
\]

**Proof.** We only prove assertion (a), leaving the other to the reader. Setting $f(t) := o_0 J_t^\mu g(t)$ and using integration by parts twice, we obtain

\[
f(t) = \frac{g(0)}{\Gamma(\mu+1)} t^\mu + \frac{g'(0)}{\Gamma(\mu+2)} t^{\mu+1} + \frac{1}{\Gamma(\mu+2)} \int_0^t (t-s)^{\mu+1} g''(s) ds.
\]  

(2.2)

It follows that the second derivative of $f$ has the form

\[
f''(t) = \frac{g(0)}{\Gamma(\mu-1)} t^{\mu-2} + \frac{g'(0)}{\Gamma(\mu)} t^{\mu-1} + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} g''(s) ds,
\]

and $g''(t)$ is bounded since $g(t) \in C^2([0, T])$. If $k \geq 1$, then he inequalities

\[
t_k^{\mu-2} \leq \frac{1}{2} t_{k+1}^{\mu-2}, \quad t_k^{\mu-1} \leq \frac{1}{2} t_{k+1}^{\mu-1}, \quad t_k^\mu \leq t_{k+1}^\mu,
\]
yield
\[ |f''(t_k)| \leq \left| \frac{g(0)}{\Gamma(\mu-1)} t_k^{\mu-2} \right| + \left| \frac{g'(0)}{\Gamma(\mu)} t_k^{\mu-1} \right| + C t_k^\mu \]
\[ \leq C \left[ |g(0)| t_k^{\mu-2} + |g'(0)| t_k^{\mu-1} + t_k^{\mu} \right] \]
\[ \leq C \left[ |g(0)| t_k^{\mu-2} + |g'(0)| t_k^{\mu-1} + t_k^{\mu} + \Gamma(\mu) \right]. \]

If \( k = 0 \), then it follows from (2.2) that
\[ f(t_{1/2}) = \frac{1}{2} f(t_1) + \mathcal{O}(\tau^\mu + |g'(0)| t_1^{\mu+1} + \tau^{\mu+2}), \]
and since \( g(0) = g'(0) = 0 \), the proof is completed. \( \square \)

**Lemma 2.3** (cf. LeVeque [18]). If \( f(t) \in C^2([t_n, t_{n+1}]) \) and \( f^{(3)}(t) \in L([t_n, t_{n+1}]) \), then
\[ f'(t_{n+1/2}) = \delta_1 f(t_{n+1/2}) + \mathcal{O}(\tau^2). \]
If \( f(t) = \mathcal{O}(t^\gamma) \) and \( 1 < \gamma < 2 \), then
\[ f'(t_{n+1/2}) = \delta_1 f(t_{n+1/2}) + \mathcal{O}\left( t_{n+1}^{\gamma-2} \right). \]

**Lemma 2.4.** If \( x \in [0, \pi] \) and
\[ \theta_x := \arcsin\left( \frac{x}{\sqrt{(3 - \cos x)^2 + \sin^2 x}} \right), \]
then \( x/2 \geq \theta_x \).

**Proof.** If \( x = 0 \) or \( x = \pi \), the inequality is obvious. Assuming that \( x \in (0, \pi) \), we only have to show that
\[ \sin\left( \frac{x}{2} \right) \geq \frac{x}{\sqrt{(3 - \cos x)^2 + \sin^2 x}} \]
and simple trigonometric formulas provide the result. \( \square \)

**Lemma 2.5.** Let \( \{\omega_k^{(\mu)}\}_{n=0}^{\infty} \) be the terms defined in Lemma 2.1. Then for any positive integer \( K \) and any real vector \( (V_1, V_2, \cdots, V_k)^T \), the inequality
\[ \sum_{n=0}^{K-1} \left( \sum_{p=0}^{n} \omega_p^{(\mu)} V_{n+1-p} \right) V_{n+1} \geq 0 \]
holds.
Proof: Using the symmetric Toeplitz matrix method and its generating function — cf. [41], we have

\[
f(\mu, x) = \omega_0^{(\mu)} + \frac{1}{2} \sum_{k=1}^{\infty} \omega_k^{(\mu)} e^{ikx} + \frac{1}{2} \sum_{k=1}^{\infty} \omega_k^{(\mu)} e^{-ikx}
\]

\[
= \frac{2^{\mu}}{2} (1 - e^{ix})^{-\mu} (3 - e^{ix})^{-\mu} + \frac{2^{\mu}}{2} (1 - e^{-ix})^{-\mu} (3 - e^{-ix})^{-\mu}
\]

\[
= \frac{1}{2} \left( \sin \frac{x}{2} \right)^{-\mu} \left[ e^{i\mu(\pi - x)/2} (3 - \cos x - i \sin x)^{-\mu} + e^{i\mu(\pi + x)/2} (3 - \cos x + i \sin x)^{-\mu} \right]
\]

\[
= \frac{1}{2} \left( \sin \frac{x}{2} \right)^{-\mu} \left[ (3 - \cos x)^2 + \sin^2 x \right]^{-\mu/2} \cos \left( \mu \frac{\pi - x}{2} + \mu \theta_x \right),
\]

where

\[
\theta_x = \arcsin \left( \frac{\sin x}{\sqrt{(3 - \cos x)^2 + \sin^2 x}} \right).
\]

Taking into account Lemma 2.4, we obtain \( \cos(\mu(\pi - x)/2 + \mu \theta_x) \geq 0 \) and, consequently, \( f(\mu, x) \geq 0 \).

Let

\[
\Delta_\beta^u(x) := \sum_{j=-\infty}^{\infty} (-1)^j \Gamma(\beta + 1) \frac{\Gamma(\beta - 2 - j + 1) \Gamma(\beta/2 + j + 1)}{\Gamma(\beta/2 - j + 1) \Gamma(\beta/2 + j + 1) \Gamma(\beta/2 + j + 1)} u(x - jh)
\]

(2.3)

be the fractional centered difference operator considered by Ortigueira [31]. The following lemma shows that this operator approximates the Riesz fractional derivative with the second-order accuracy.

Lemma 2.6 (cf. Celik and Duman [4]). Let \( 1 < \beta \leq 2 \). If \( u(x) \in C^5(\mathbb{R}) \) and all its derivatives up to the order five belong to \( L(\mathbb{R}) \), then

\[
\frac{\partial^\beta u(x)}{\partial |x|^\beta} = -\frac{\Delta_\beta^u(x)}{h^\beta} + \Theta(h^2).
\]

Note that if the support of \( u(x) \) is in a finite interval \([0, L]\), then

\[
\frac{\partial^\beta u(x)}{\partial |x|^\beta} = -\delta_x^\beta u(x) + \Theta(h^2)
\]

with the operator \( \delta_x^\beta \) defined by

\[
\delta_x^\beta u(x) := \frac{1}{h^\beta} \sum_{j=-\lfloor (\beta - 1)/2 \rfloor}^{\lfloor (\beta - 1)/2 \rfloor} (-1)^j \Gamma(\beta + 1) \frac{\Gamma(\beta/2 - j + 1) \Gamma(\beta/2 + j + 1)}{\Gamma(\beta/2 - j + 1) \Gamma(\beta/2 + j + 1)} u(x - jh).
\]

Lemma 2.7 (cf. Wang & Huang [40]). For \( 1 < \beta < 2 \) and the operator \( \delta_x^\beta \), there exists a linear difference operator, denoted by \( \delta_x^{\beta/2} \), such that

\[
\langle \delta_x^\beta U, V \rangle = \langle \delta_x^{\beta/2} U, \delta_x^{\beta/2} V \rangle,
\]

where \( U, V \in \Omega_h \).
3. Convolution Quadrature Method 1

Assume that \( u(\cdot,t) = \mathcal{O}(t^\gamma) \) with \( 1 < \gamma < 2 \) and \( u(x,\cdot) \in C^3([0,L]) \) with homogenous boundary values. Let \( u^n_j \) and \( U^n_i \) be, respectively, exact and numerical solutions at the mesh point \((x_j,t_n)\). Consider the integro-differential equation (2.1) at the point \((x_i,t_{n+1/2})\) — i.e.

\[
\frac{\partial u(x_i,t)}{\partial t} \bigg|_{t=t_{n+1/2}} = K_c \cdot \mathcal{O}^{(a-1)} \left[ \frac{\partial^\beta u(x_i,t)}{\partial|x|^\beta} \right] + \mathcal{O}^{(a-1)} \left[ g(u) \right] + F_{i}^{n+1/2}, \tag{3.1}
\]

where \( F_{i}^{n+1/2} = F(x_i,t_{n+1/2}) \).

The Crank-Nicolson technique and Lemma 2.2 yield

\[
\frac{\partial u(x_i,t)}{\partial t} \bigg|_{t=t_{n+1/2}} = \frac{K_c}{2} \left[ \mathcal{O}^{(a-1)} \frac{\partial^\beta u(x_i,t)}{\partial|x|^\beta} + \mathcal{O}^{(a-1)} \frac{\partial^\beta u(x_i,t)}{\partial|x|^\beta} \right] \\
+ \frac{1}{2} \left[ \mathcal{O}^{(a-1)} g(u) + \mathcal{O}^{(a-1)} g(u) \right] \\
+ F_{i}^{n+1/2} + \mathcal{O} \left( t_{n+1}^{\alpha-3} + \tau^2 \right). 
\]

Note that we used the central differences (2.3) to discretise the first-order derivative. The Riesz fractional derivative and the Riemann-Liouville fractional integral in the right-hand side of (3.1) are approximated according to Lemmas 2.6 and 2.1— i.e.

\[
\delta_t u_i^{n+1/2} = \frac{-K_c \tau^{a-1}}{2} \left[ \sum_{k=0}^{n+1} \omega_k (a-1) \delta_x^\beta u_i^{n+1-k} + \sum_{k=0}^{n} \omega_k (a-1) \delta_x^\beta U_i^{n-k} \right] \\
+ \frac{\tau^{a-1}}{2} \left[ \sum_{k=0}^{n+1} \omega_k (a-1) g(U_i^{n+1-k}) + \sum_{k=0}^{n} \omega_k (a-1) g(U_i^{n-k}) \right] \\
+ F_{i}^{n+1/2} + (R_1)_i^{n+1}, \tag{3.2}
\]

where

\[
(R_1)_i^{n+1} = \mathcal{O} \left( t_{n+1}^{\gamma-3} + \tau^2 + t_{n+1}^{\alpha-3} + h^2 \right) = \mathcal{O} \left( t_{n+1}^{\gamma-3} + \tau^2 + h^2 \right). 
\]

Neglecting the truncation error term in (3.2), we arrive at the convolution quadrature method, referred to as Method 1. Thus

\[
U_i^{n+1} - U_i^n = \frac{-K_c \tau^{a}}{2} \left[ \sum_{k=0}^{n+1} \omega_k (a-1) \delta_x^\beta U_i^{n+1-k} + \sum_{k=0}^{n} \omega_k (a-1) \delta_x^\beta U_i^{n-k} \right] \\
+ \frac{\tau^{a}}{2} \left[ \sum_{k=0}^{n+1} \omega_k (a-1) g(U_i^{n+1-k}) + \sum_{k=0}^{n} \omega_k (a-1) g(U_i^{n-k}) \right] \\
+ \tau F_{i}^{n+1/2}, \tag{3.3}
\]

or more clearly as

\[
\left( 1 + \frac{K_c \tau^{a}}{2} g_0^{(a-1)} \right) U_i^{n+1} = U_i^n - \frac{K_c \tau^{a}}{2} \sum_{k=0}^{n} \left( \omega_k (a-1) + \omega_k (a-1) \right) \delta_x^\beta U_i^{n-k}
\]
Suppose that \( \| e \| \) where Lemma 2.7 leads to the representation

\[
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\]

It is clear that this method requires finding solutions of the system of nonlinear equations at each time level. This can be done by standard methods — e.g. by the Newton method.

**Theorem 3.1.** Assume that \( u(\cdot, t) = O(t^\gamma), \ 1 < \gamma < 2 \) and \( u(x, \cdot) \in C^5([0, L]) \) with \( u(0, \cdot) = u(L, \cdot) = 0 \). Then for sufficiently small \( \tau \), Method 1 converges and

\[
\| e^n \| = O(\tau^\gamma + h^2), \quad n = 1, 2, \ldots, N.
\]

**Proof.** Subtracting (3.2) from (3.3) implies that

\[
e^{n+1}_i - e^n_i = -\frac{K \tau^a}{2} \left[ \sum_{k=0}^{n+1} \omega_k^{(a-1)} \delta_x^{(a-1)} n_{n-k} + \sum_{k=0}^{n} \omega_k^{(a-1)} \delta_x^{(a-1)} n_{n-k} \right]
\]

\[
+ \frac{\tau^a}{2} \sum_{k=0}^{n+1} \omega_k^{(a-1)} \left[ g(U^{n+1-k}) - g(U^n) \right]
\]

\[
+ \frac{\tau^a}{2} \sum_{k=0}^{n} \omega_k^{(a-1)} \left[ g(U^{n-k}) - g(U^n) \right] + \tau R_i^{n+1},
\]

(3.4)

where \( e^i = u^i - u_i^j \). Multiplying both sides of (3.4) by \( h(e^{n+1}_i + e^n_i) \) and summing the results in \( i \), we obtain

\[
\langle e^{n+1} - e^n, e^{n+1} + e^n \rangle = -\frac{K \tau^a}{2} \sum_{k=0}^{n} \omega_k^{(a-1)} \langle \delta_x^{(a-1)} (e^{n+1-k} + e^{n-k}), e^{n+1} + e^n \rangle
\]

\[
+ \frac{\tau^a}{2} \sum_{k=0}^{n+1} \omega_k^{(a-1)} \langle [g(U^{n+1-k}) - g(U^{n+1-k})], e^{n+1} + e^n \rangle
\]

\[
+ \frac{\tau^a}{2} \sum_{k=0}^{n} \omega_k^{(a-1)} \langle [g(U^{n-k}) - g(U^{n-k})], e^{n+1} + e^n \rangle
\]

\[
+ \langle \tau R_i^{n+1}, e^{n+1} + e^n \rangle.
\]

(3.5)

Suppose that \( \| e^{K+1} \| = \max_{1 \leq n \leq N} \| e^n \| \). Summing the Eq. (3.5) in \( n \) from 0 to \( K \) and using Lemma 2.7 leads to the representation

\[
\| e^{K+1} \|^2 = -\frac{K \tau^a}{2} \sum_{n=0}^{K} \sum_{k=0}^{n} \omega_k^{(a-1)} \langle \delta_x^{(a-1)} (e^{n+1-k} + e^{n-k}), \delta_x^{(a-1)} (e^{n+1} + e^n) \rangle
\]

\[
+ \frac{\tau^a}{2} \sum_{n=0}^{K} \omega_n^{(a-1)} \langle [g(U^{n+1}) - g(U^{n+1})], e^{n+1} + e^n \rangle
\]
Moving the term $L^1$ if the influence of the first temporal step errors weakens. Thus, we can deduce that at $t_1$ and (3.8) yields

$$+ \tau \sum_{n=0}^{K} (\tau (R_1)^{n+1} + e^{n+1} + \varepsilon).$$

(3.6)

It follows from Lemma 2.5 that the first term in the right-hand side of (3.6) is negative and since $g$ satisfies the Lipschitz condition with a constant $L_p$, the norm $\|e^{K+1}\|^2$ can be estimated as

$$\|e^{K+1}\|^2 \leq L_p \tau^a \omega_0^{(a-1)} \|e^{K+1}\| + L_p \tau^a \sum_{n=0}^{K} \omega_0^{(a-1)} ||e^n||$$

$$+ L_p \tau^a \sum_{n=0}^{K} \sum_{k=0}^{n} (\omega_0^{(a-1)} + \omega_k^{(a-1)}) ||e^{n-k}|| + C \sum_{n=0}^{K} (t_{n+1}^{\gamma-3} + \tau h^2).$$

(3.7)

Moving the term $L_p \tau^a \omega_0^{(a-1)} ||e^{K+1}\|$ to the left-hand side of (3.7), and noting that $\sum_{n=0}^{K} (n + 1)^{\gamma-3}$ is convergent, we obtain that for sufficiently small $\tau$ the inequality

$$\|e^{K+1}\| \leq C (\tau^\gamma + h^2) + C \tau^a \sum_{n=0}^{K} \omega_0^{(a-1)} ||e^n|| + C \tau^a \sum_{n=0}^{K} \sum_{k=0}^{n} (\omega_0^{(a-1)} + \omega_k^{(a-1)}) ||e^{n-k}||$$

(3.8)

holds. Since $\omega_k^{(a-1)} = \mathcal{O}(k^{a-2})$ — cf. [23], the sums $\tau^{a-1} \sum_{n=0}^{K} (\omega_0^{(a-1)} + \omega_k^{(a-1)})$ are bounded and (3.8) yields

$$\|e^{K+1}\| \leq C (\tau^\gamma + h^2) + C \tau \sum_{n=0}^{K} ||e^n||.$$ 

(3.9)

Applying the Gronwall inequality to (3.9), we arrive at the estimate

$$\|e^{K+1}\| \leq C (\tau^\gamma + h^2),$$

thus finishing the proof. □

**Remark 3.1.** Although Theorem 3.1 shows that Method 1 has accuracy $\mathcal{O}(\tau^\gamma + h^2)$, this estimate is too conservative. The local truncation error for the Eq. (3.2) is $\mathcal{O}(t_{n+1}^{\gamma-3} \tau^2 + h^2)$, but if $t$ is far away from $t_0$, local truncation error in temporal direction can be $\mathcal{O}(\tau^2)$. Besides, if $t$ grows, the weight coefficients $\omega_k^{(a-1)}$ in the first temporal steps decrease and the influence of the first temporal step errors weakens. Thus, we can deduce that at $t = t_0$, Method 1 has accuracy $\mathcal{O}(\tau^\gamma + h^2)$ and can become $\mathcal{O}(\tau^2 + h^2)$ when $t_{n+1}$ is far away from $t_0$. The results of numerical experiments in Section 5 are consistent with this remark.

**Theorem 3.2.** If $\tau$ is sufficiently small, then under conditions of Theorem 3.1, Method 1 is absolutely stable — viz.

$$\|\eta^{K+1}\| \leq C (\|\eta^0\| + \tau \|\delta_x \eta^0\| + \max_{0 \leq n \leq K} ||F_n^{\eta+1/2} - \bar{F}_n^{\eta+1/2}||).$$

(3.10)
Theorem 3.3. If \( \mathcal{F} \) is sufficiently small, \( u^n(t) \) is the solution of the difference equation

\[
\tilde{U}_i^{n+1} - \tilde{U}_i^n = -\frac{K_i \tau}{2} \sum_{k=0}^{n+1} \omega_k^{(a-1)\beta} \tilde{U}_i^{n+1-k} + \sum_{k=0}^{n} \omega_k^{(a-1)\beta} \tilde{U}_i^{n-k} \\
+ \frac{\tau^a}{2} \sum_{k=0}^{n+1} \omega_k^{(a-1)} g \left( \tilde{U}_i^{n+1-k} \right) + \sum_{k=0}^{n} \omega_k^{(a-1)} g \left( \tilde{U}_i^{n-k} \right) \\
+ \tau \tilde{F}_i^{n+1/2},
\]

(3.11)

we write \( \eta_i^n = U_i^n - \tilde{U}_i^n \) and subtracting (3.11) from (3.3) yields

\[
\eta_i^{n+1} - \eta_i^n = -\frac{K_i \tau}{2} \sum_{k=0}^{n+1} \omega_k^{(a-1)\beta} \eta_i^{n+1-k} + \sum_{k=0}^{n} \omega_k^{(a-1)\beta} \eta_i^{n-k} \\
+ \frac{\tau^a}{2} \sum_{k=0}^{n+1} \omega_k^{(a-1)} \left[ g \left( U_i^{n+1-k} \right) - g \left( \tilde{U}_i^{n+1-k} \right) \right] \\
+ \sum_{k=0}^{n} \omega_k^{(a-1)} \left[ g \left( U_i^{n-k} \right) - g \left( \tilde{U}_i^{n-k} \right) \right] \\
+ \tau \left[ F_i^{n+1/2} - \tilde{F}_i^{n+1/2} \right].
\]

Following the approach from the proof of Theorem 3.1, we obtain the estimate (3.10).

**Theorem 3.3.** If \( \tau \) is sufficiently small, \( u(\cdot, t) \in C^2([0, T]) \), \( u(\cdot, 0) = u_0(\cdot, 0) = 0 \) and \( u(x, \cdot) \in C^2([0, L]) \), \( u(0, \cdot) = u(L, \cdot) = 0 \), then the Method 1 is absolutely stable and

\[
\|e^n\| = \Theta(\tau^2 + h^2), \quad n = 1, 2, \ldots, N.
\]

4. Convolution Quadrature Method 2

Ding et al. [9] proposed a second-order midpoint approximation formula for the Riemann-Liouville derivative at time \( t_{n+1/2} \), which is well suited to the Crank-Nicolson scheme for time fractional differential equations. Aiming to reduce the smoothness requirement of this approximation and to improve the numerical stability, we consider a second-order midpoint approximation formula for Riemann-Liouville integral at time \( t_{n-1/2} \).

**Lemma 4.1** (cf. Podlubny [32]). Let

\[
\mathcal{F} \left( -\infty J_t^\mu u(t) \right) = (i\omega)^{-\mu} \tilde{u}(\omega),
\]

where \( \tilde{u}(\omega) \) is the Fourier transform of \( u(t) \).
\textbf{Theorem 4.1.} Let $0 < \mu < 1$. If $u(t) \in C^2(\mathbb{R})$ and $u^{(3)}(t) \in L_1(\mathbb{R})$, then

\begin{equation}
-\infty \int^0_{t} u \left( t - \frac{1}{2} \tau \right) = \tau^\mu \sum_{k=0}^{\infty} \tilde{\omega}_k^{(\mu)} u(t - k\tau) + o(\tau^2),
\end{equation}

where $\tilde{\omega}_k^{(\mu)}$, $k = 0, 1, \cdots$ are the weights determined by the function $G(z)$,

\begin{equation}
G(z) = \left( \frac{3\mu + 1}{2\mu} - \frac{2\mu + 1}{\mu}z + \frac{\mu + 1}{2\mu}z^2 \right)^{-\mu} = \sum_{k=0}^{\infty} \tilde{\omega}_k^{(\mu)} z^k.
\end{equation}

\textbf{Proof.} Applying the Fourier transform to the function $\tau^\mu \sum_{k=0}^{\infty} \tilde{\omega}_k^{(\mu)} u(t - k\tau)$ and using (4.2), we obtain

\begin{align*}
\mathcal{F} \left\{ \tau^\mu \sum_{k=0}^{\infty} \tilde{\omega}_k^{(\mu)} u(t - k\tau) \right\} (\omega) &= \tau^\mu \sum_{k=0}^{\infty} \tilde{\omega}_k^{(\mu)} e^{-i\omega\tau} \hat{u}(\omega) \\
&= \tau^\mu \hat{u}(\omega) G(e^{-\omega}) \\
&= (i\omega)^{-\mu} e^{-i\omega\tau/2} S(i\omega\tau) \hat{u}(\omega),
\end{align*}

where $S(z) = z^\mu e^{z/2} G(e^{-z})$. The Taylor's expansion of the function $G(e^{-z})$ has the form

\begin{align*}
G(e^{-z}) &= \left( \frac{3\mu + 1}{2\mu} - \frac{2\mu + 1}{\mu}z + \frac{\mu + 1}{2\mu}z^2 \right)^{-\mu} \\
&= z^{-\mu} \left[ 1 + \frac{1}{2\mu}z - \frac{2\mu + 3}{6\mu}z^2 + o(z^3) \right]^{-\mu} \\
&= z^{-\mu} \left[ 1 + \mu \left( \frac{1}{2\mu}z + \frac{2\mu + 3}{6\mu}z^2 + \frac{\mu + 1}{2}\left( -\frac{1}{2\mu}z + \frac{2\mu + 3}{6\mu}z^2 \right)^2 + o(z^3) \right) \right] \\
&= z^{-\mu} \left[ 1 - \frac{1}{2} z^2 + \frac{8\mu^2 + 15\mu + 3}{24\mu} z^2 + o(z^3) \right],
\end{align*}

so that

\begin{align*}
S(z) &= z^\mu e^{z/2} G(e^{-z}) \\
&= z^\mu \left( 1 + \frac{z}{2} + \frac{z^2}{8} + o(z^3) \right) z^{-\mu} \left[ 1 - \frac{1}{2} z^2 + \frac{8\mu^2 + 15\mu + 3}{24\mu} z^2 + o(z^3) \right] \\
&= 1 - \frac{8\mu^2 + 12\mu + 3}{24\mu} z^2 + o(z^3).
\end{align*}

Thus

\begin{align*}
\mathcal{F} \left\{ \tau^\mu \sum_{k=0}^{\infty} \tilde{\omega}_k^{(\mu)} u(t - k\tau) \right\} (\omega) &= \mathcal{F} \left\{ -\infty \int^0_{t} u \left( t - \frac{1}{2} \tau \right) \right\} + \phi(\tau, \omega),
\end{align*}
where \( \hat{\phi}(\tau, \omega) = (i\omega)^{-\mu}e^{-i\omega\tau/2}(S(i\omega\tau)-1)\hat{u}(\omega) \). Since \( u(t) \in C^2(\mathbb{R}) \) and \( u^{(3)}(t) \in L_1(\mathbb{R}) \), the estimate

\[
|\hat{u}(\omega)| \leq C(1 + |\omega|)^{-3}
\]

holds. Therefore,

\[
|\phi(\tau, t)| = \left| \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{i\omega\tau} \hat{\phi}(\tau, \omega) d\omega \right| \leq C \int_{-\infty}^{+\infty} (1 + |\omega|)^{2-\mu} |\hat{u}(\omega)| d\omega = \mathcal{O}(\tau^2),
\]

and the relation (4.1) follows.

**Corollary 4.1.** Let \( 0 < \mu < 1 \). If \( u(t) \in C^2([0, T]) \), \( u^{(3)}(t) \in L_1([0, T]) \) and \( u(t) = 0, t \in (-\infty, 0] \), then

\[
0^J_t \mu \left( t_n - \frac{1}{2} \tau \right) = \tau^{\mu} \sum_{k=0}^{n} \hat{\omega}^{(n)}_k u(t_n - k\tau) + \mathcal{O}(\tau^2), \quad n = 1, 2, \ldots.
\]  

(4.3)

Assume that \( u(\cdot, t) \in C^2([0, T]), u(\cdot, 0) = u_1(\cdot, 0) = 0, u(x, \cdot) \in C^2([0, L]), u(0, \cdot) = u(L, \cdot) = 0 \) and consider the Eq. (2.1) at the point \((x_i, t_{n-1/2})\) — i.e.

\[
\frac{\partial u(x_i, t_{n-1/2})}{\partial t} = \frac{K_c}{2} \left[ 0^J_{t_n} \frac{\partial^\beta u(x_i, t)}{\partial x^\beta} + 0^J_{t_{n-1}} \frac{\partial^\beta u(x_i, t)}{\partial x^\beta} \right] + 0^J_{t_{n-1/2}} [g(u)] + F_i(x_i, t_{n-1/2}).
\]

The Crank-Nicolson technique and Lemma 2.2 yield that

\[
\frac{\partial u(x_i, t_{n-1/2})}{\partial t} = \frac{K_c}{2} \left[ 0^J_{t_n} \frac{\partial^\beta u(x_i, t)}{\partial x^\beta} + 0^J_{t_{n-1}} \frac{\partial^\beta u(x_i, t)}{\partial x^\beta} \right] + 0^J_{t_{n-1/2}} [g(u)] + \mathcal{O}(\tau^2).
\]

(4.4)

Using now Lemmas 2.1, 2.3 and Corollary 4.1 to discretise the Eq. (4.4), we write

\[
\delta_t u_i^{n-1/2} = -\frac{K_c}{2} \sum_{k=0}^{n} \omega_k^{(a-1)} \frac{\partial^\beta_x u_i^{n-k}}{\partial x^\beta} + \sum_{k=0}^{n-1} \omega_k^{(a-1)} \frac{\partial^\beta_x u_i^{n-1-k}}{\partial x^\beta} + \tau^{a-1} \sum_{k=0}^{n} \omega_k^{(a-1)} g(u_i^{n-k}) F_i^{n-1/2} + \mathcal{O}(\tau^2 + h^2).
\]

Neglecting the truncation error term, we arrive at the second convolution quadrature method — viz.

\[
\left( 1 + \frac{K_c \tau^a}{2} \omega_0^{(a-1)} \frac{\partial^\beta_x}{\partial x^\beta} \right) U_i^n = U_i^{n-1} - \frac{K_c \tau^a}{2} \sum_{k=0}^{n-1} \left( \omega_k^{(a-1)} + \omega_k^{(a-1)} \right) \frac{\partial^\beta_x u_i^{n-1-k}}{\partial x^\beta} + \tau^{a} \sum_{k=0}^{n} \omega_k^{(a-1)} g(u_i^{n-k}) + \tau F_i^{n-1/2}.
\]

In what follows it is referred to as Method 2.
Remark 4.1. The representation (4.3) is not directly used for the discretisation of the term \(0^{\frac{\alpha-1}{2}}_t \int \frac{\partial^\beta u(x,t)}{\partial |x|^{\beta}}\) since the form

\[
\sum_{n=0}^{N-1} \left( \sum_{p=0}^{n} \omega_k^{(n-1)} V_{n+1-p} \right) V_{n+1}
\]

is not positive definite for all \(\alpha \in (1,2)\).

Actually, using the approach from the proof of Lemmas 2.4 and 2.5, we obtain the inequality \(2\alpha/(\alpha+1) \geq 2\) which is wrong for \(\alpha > 0\).

Analogously to Theorems 3.1 and 3.2, one can prove the following results.

Theorem 4.2. If \(\tau\) is sufficiently small, \(u(\cdot,t) \in C^2([0,T])\), \(u(\cdot,0) = u_0\), \(u_t(\cdot,0) = 0\) and \(u(x,\cdot) \in C^5([0,L])\), \(u(0,\cdot) = u(L,\cdot) = 0\), then Method 2 is absolutely stable and

\[
\|e^n\| = O(\tau^2 + h^2), \quad n = 1, 2, \ldots, N.
\]

5. Numerical Experiments

We present two numerical examples to support theoretical findings and demonstrate the efficiency of the methods proposed.

Example 5.1. Consider the time-space fractional nonlinear diffusion-wave equation

\[\frac{\partial^\beta}{\partial |x|^{\beta}} u(x,t) = \frac{\partial^\beta u(x,t)}{\partial |x|^{\beta}} + f(x,t) + \sin(u(x,t)), \quad 0 < x < 1, \quad 0 < t \leq 1\]  (5.1)

with homogeneous initial-boundary conditions and the exact solution \(u(x,t) = t^{1+\alpha}x^2(1-x)^2\). In this case, the right-hand side \(f(x,t)\) is the sum of the linear function

\[
f_t(x,t) = \Gamma(2+\alpha)t x^2(1-x)^2 + \frac{t^{1+\alpha}}{\cos((\pi \beta/2)\Gamma(5-\beta))} \times \left( 12 \left[x^{4-\beta} + (1-x)^{4-\beta}\right] - 6(4-\beta) \left[x^{3-\beta} + (1-x)^{3-\beta}\right] + (3-\beta)(4-\beta) \left[x^{2-\beta} + (1-x)^{2-\beta}\right] \right),
\]

and the nonlinear function \((-\sin(t^{1+\alpha}x^2(1-x)^2))\). Reducing the Eq. (5.1) to equivalent partial integro-differential equation, we obtain

\[
u_t(x,t) = \frac{\partial^\beta}{\partial |x|^{\beta}} u(x,t) + f_t(x,t) + \frac{\partial^\beta}{\partial |x|^{\beta}} \sin(u(x,t))
\]

\[
- \frac{\partial^\beta}{\partial |x|^{\beta}} \sin \left(t^{1+\alpha}x^2(1-x)^2\right), \quad (5.2)
\]

where \(f_t(x,t)\) is explicitly computable. The term \(\frac{\partial^\beta}{\partial |x|^{\beta}} \sin(t^{1+\alpha}x^2(1-x)^2)\) is approximated according to Lemma 2.1.
It is clear that \( u(x, t) \) satisfies all smoothness conditions required by Methods 1 and 2, so that both of them can be applied to the Eq. (5.2). To check the convergence order in time, we take the step size \( h = 0.001 \) and compute \( L_2 \)-norm errors at \( T = 1 \) for methods with different temporal step size \( \tau \). Tables 1 and 2 show the errors for \( \beta = 1.5 \) and various \( \alpha \). Note that the errors decrease along with the temporal mesh size and for both methods numerical convergence order is 2, consistent with theoretical results for solutions from \( C^2([0, 1]) \). We also observe that Method 2 has a better accuracy than the other one.

To check the convergence order in space, we choose temporal mesh size \( \tau = 0.001 \) and vary spatial step size. Table 3 shows that spatial convergence order for Methods 1 is equal to 2, consistent with the theoretical analysis. Method 2 behaves similar, so we omit the results here.

Table 1: Example 5.1. Errors and temporal numerical convergence order for Method 1. \( \beta = 1.5, h = 0.001 \).

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( \alpha = 1.25 )</th>
<th>( \alpha = 1.5 )</th>
<th>( \alpha = 1.75 )</th>
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<tr>
<td>error</td>
<td>order</td>
<td>error</td>
<td>order</td>
</tr>
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<tr>
<td>1/80</td>
<td>3.5498e-6</td>
<td>1.9974</td>
<td>9.2793e-6</td>
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<tr>
<td>1/160</td>
<td>8.6569e-7</td>
<td>2.0131</td>
<td>2.3171e-6</td>
</tr>
</tbody>
</table>

Table 2: Example 5.1. Errors and temporal numerical convergence order for Method 2. \( \beta = 1.5, h = 0.001 \).

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( \alpha = 1.25 )</th>
<th>( \alpha = 1.5 )</th>
<th>( \alpha = 1.75 )</th>
</tr>
</thead>
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<tr>
<td>error</td>
<td>order</td>
<td>error</td>
<td>order</td>
</tr>
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</tbody>
</table>

Table 3: Example 5.1. Errors and spatial numerical convergence order for Method 1. \( \alpha = 1.5, \tau = 0.001 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \beta = 1.25 )</th>
<th>( \beta = 1.5 )</th>
<th>( \beta = 1.75 )</th>
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<td>order</td>
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Example 5.2. We now consider the time-space fractional nonlinear diffusion-wave equation

\[ C_D^\alpha u(x,t) = \frac{\partial^\beta u(x,t)}{\partial|x|^\beta} + f(x,t) + u^2(x,t), \quad 0 < x < 1, \quad 0 < t \leq 1 \quad (5.3) \]

with homogeneous initial-boundary conditions and with the solution \( u(x,t) = 5t^\gamma x^2(1-x)^2 \). The corresponding linear function \( f(x,t) \) has the form

\[
 f(x,t) = \frac{5\Gamma(\gamma + 1)}{\Gamma(\gamma - \alpha + 1)} t^{\gamma-\alpha}x^2(1-x)^2 + \frac{5t^\gamma}{\cos(\pi\beta/2)\Gamma(5-\beta)} \\
 \times \left( 12\left[x^{4-\beta} + (1-x)^4-\beta \right] - 6(4-\beta)\left[x^{3-\beta} + (1-x)^3-\beta \right] \\
 + (3-\beta)(4-\beta)\left[x^{2-\beta} + (1-x)^2-\beta \right] \right) - 25t^{2\gamma}x^4(1-x)^4.
\]

Note that \( u(\cdot,t) = \mathcal{O}(t^\gamma), \, 1 < \gamma < 2 \), so that the only Method 1 can be used.

The Eq. \((5.3)\) can be reduced to an equivalent partial integro-differential equation — viz.

\[
 u_t(x,t) = \frac{\partial^\beta u(x,t)}{\partial|x|^\beta} + F(x,t) + \mathcal{O}^{a-1}u^2(x,t)
\]

with explicitly computable function \( F(x,t) \).

For \( \alpha = 1.4, \beta = 1.7 \) and \( \gamma = 1.6 \), the approximate and exact solutions of this equation at \( T = 1 \) are displayed in Fig. 1. Analytical and numerical results are in excellent agreement.

To verify Theorem 3.1 and Remark 3.1, we compute the \( L_2 \)-errors and the temporal numerical convergence orders at the first \( t_1 \) and final \( t_N \) time steps for \( \alpha = \beta = 1.5 \) and various \( \gamma \) — cf. Tables 4 and 5. It is clearly visible that at \( t_1 \) the numerical convergence

![Graph](image-url)
order approaches $\gamma$ and at $t_N$ it is close to 2. This is consistent with the theoretical analysis.

Interestingly, for $\gamma = 1.3$, $\gamma = 1.5$ and small $\tau$, the error in the first step is larger than in the final one.

Table 6 contains spatial $L_2$-errors and convergence orders for fixed $\alpha$ and $\gamma$ and changing $\beta$. The convergence rates approach 2, consistent with the theoretical results again.

**Table 4:** Example 5.2. Errors and temporal numerical convergence order for Method 1 at $t_1$, $\alpha = 1.5$, $\beta = 1.5$, $h = 0.001$.

<table>
<thead>
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<th>$\tau$</th>
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<th>$\gamma = 1.5$</th>
<th></th>
<th>$\gamma = 1.8$</th>
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<td>3.6227e-5</td>
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<td>7.0120e-7</td>
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**Table 5:** Example 5.2. Errors and temporal numerical convergence order for Method 1 at $t_N$, $\alpha = 1.5$, $\beta = 1.5$, $h = 0.001$.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\gamma = 1.3$</th>
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<th>$\gamma = 1.5$</th>
<th></th>
<th>$\gamma = 1.8$</th>
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<tbody>
<tr>
<td></td>
<td>error</td>
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<td>error</td>
<td>order</td>
<td>error</td>
<td>order</td>
</tr>
<tr>
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<td>2.7230e-3</td>
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<td>4.6484e-3</td>
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<tr>
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<td>6.6916e-4</td>
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<td>1.1740e-3</td>
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</table>

**Table 6:** Example 5.2. Errors and spatial numerical convergence order for Method 1. $\alpha = 1.3$, $\gamma = 1.7$, $\tau = 0.001$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\beta = 1.3$</th>
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<th>$\beta = 1.6$</th>
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<th>$\beta = 1.9$</th>
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<tbody>
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<td>order</td>
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<tr>
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<td>1.5836e-2</td>
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<tr>
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<td>2.1362</td>
<td>6.4719e-3</td>
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6. Concluding Remarks

We constructed two second-order convolution quadrature methods for fractional nonlinear diffusion-wave equations with Caputo derivative in time and Riesz derivative in space. The equations are reduced to equivalent partial integro-differential equations and the classical Crank-Nicolson technique and the second-order convolution quadratures based on the generating function \( (3/2 - 2z + z^2/2)^{\mu}, 0 < \mu < 1 \) are used. The methods work well with time low regularity solutions and provide second-order accuracy. If the solution is twice differentiable in time, a new second-order convolution quadrature method for the Riemann-Liouville integral at \( t_{k-1/2} \) is proposed. It reduces the computational complexity if the Crank-Nicolson technique is used. The spatial Riesz fractional derivative is discretised by fractional centered differences. Theoretical and numerical approaches demonstrate unconditional stability of the methods and convergence with accuracy \( O(\tau^2 + h^2) \).

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References


