East Asian Journal on Applied Mathematics Vol. **9**, No. 3, pp. 409-423
doi: 10.4208/eajam.121218.180419 August 2019

The Breakdown of Darboux's Principle and Natural Boundaries for a Function Periodised from a Ramanujan Fourier Transform Pair

John P. Boyd*

Department of Climate and Space Sciences and Engineering, University of Michigan, 2455 Hayward Avenue, MI 48109-2143, USA.

Received 12 December 2018; Accepted (in revised version) 18 April 2019.

Abstract. Darboux's Principle asserts that a power series or Fourier coefficient a_n for an analytic function $f(z)$ is approximated as $n \to \infty$ by a sum of terms, one for each singularity of $f(z)$ in the complex plane. This is crucial to understanding why Fourier series converge rapidly or slowly, and thus crucial to Fourier numerical methods. We partially refute Darboux's Principle by an explicit counterexample constructed by applying the Poisson Summation Theorem to a Fourier Transform pair found explicitly by Ramanujan. The Fourier coefficients show a geometric rate of decay proportional to $\exp(-\pi\chi\, n)$ multiplied by $\sin(\varphi)$ where the "phase" is $\varphi = \pi\chi^2 n^2 \mod (2\pi)$. We prove that the Fourier series converges everywhere within the largest strip centered on the real axis which is singularity-free, here $|\Im(z)| < \pi\chi$. We present strong evidence that the boundaries of the strip of convergence are natural boundaries. Because the function $f(z)$ is singular everywhere on the lines $\Im(z) = \pm\pi\chi$, there is no simple way to extrapolate the asymptotic form of the Fourier coefficients from knowledge of the singularities, as is possible through Darboux's Theorem when the singularities are isolated poles or branch points.

AMS subject classifications: 65M70, 30D99, 42A38

Key words: Darboux' Theorem, Fourier Transform, imbricate, Ramanujan, natural boundary.

1. Introduction

Fourier and Chebyshev polynomial spectral methods have become very popular in many fields of science and engineering [6–8, 10, 12, 13, 17–19, 24, 29, 30]. Shen, Tang and Wang [27] write "Along with finite differences and finite elements, spectral methods are one of the three main methodologies for solving partial differential equations on computers". Many practical issues arise in spectral methods, but the most fundamental is how rapidly or slowly the Fourier series converge. This makes convergence theory important to applied mathematics as well as being an ancient and beautiful area of pure mathematics.

*Corresponding author. *Email addresses:* `jpboyd@umich.edu` (J.P. Boyd)

In passing note that the identity $T_n(\cos(t)) = \cos(nt)$ implies that a Chebyshev polynomial series is a Fourier cosine series with a change of variable, and therefore Chebyshev theory follows almost trivially from Fourier rate-of-convergence theory.

Much is known as summarised in the author's book [7] and Trefethen's monograph [30]. Much is still to be learned by studying interesting examples; broad theorems and numerical heuristics are solid only if built on a foundation of particular instances and illustrations. Here, we discuss examples that defied our expectations, the intuition from existing theory. In the process, we find a modern use for some neglected work of Ramanujan [1,21].

Fourier Transform theory, Darboux's Principle, and the Poisson Summation Theorem in its general imbricate series form are key technologies. Fear not if some of these are unfamiliar; we shall provide the necessary background as we shall go. Darboux's Principle is that each singularity of a function $f(z)$, and a similar factor for entire (integral) functions, makes its own unique and readily-calculable contribution to the asymptotic Fourier coefficients of the series for $f(z)$. The Poisson Summation Theorem as used here shows that every periodic function $f(z)$ can be represented as an infinite series ("imbricate series") of uniformly spaced copies of a "pattern function" or "imbrex" which is the Fourier Transform of the function $g(n)$ that gives the Fourier coefficients of $f(z)$.

Ramanujan published some novel Fourier integrals in 1915 and 1919 [25,26]. Watson wrote a second sequel in 1936 [32]. Titchmarsh reviewed their contributions in his 1937 book on Fourier integrals [28].

Ramanujan discovered four transform pairs, but we shall present only the following which is typical:

Lemma 1.1 (Ramanujan's Fourier Transform Pair). *The function $G(z)$ defined below is the Fourier Transform of $g(n)$, where*

$$g(n) = \sin(\pi n^2)\,\mathrm{sech}\,(\pi n), \quad G(z) = \left\{\cos\left(\frac{z^2}{4\pi}\right) - \frac{1}{\sqrt{2}}\right\}\mathrm{sech}\,(z/2)$$

and the Fourier Transform is normalised so that

$$\int_{-\infty}^{\infty} \sin(\pi z^2)\,\mathrm{sech}\,(\pi z)\exp(ikz)\,dz = \left\{\cos\left(\frac{k^2}{4\pi}\right) - \frac{1}{\sqrt{2}}\right\}\mathrm{sech}\,(k/2).$$

Ramanujan and Watson went no further in the direction pursued here. However, substituting this pair into the Poisson Theorem [5,11,35] allows us to create an analytic periodic function of period P with the property that the Fourier coefficients $g(n)$ are given exactly by one of Ramanujan's functions and the pattern function that generates the second series below, is the Fourier Transform of the Fourier coefficient function $g(n)$,

$$\begin{aligned}\mathfrak{F}(z;\chi) &\equiv \chi \sum_{n=-\infty}^{\infty} \sin(\pi\chi^2 n^2)\,\mathrm{sech}\,(\pi\chi n)\exp\left(i\frac{2\pi}{P}nz\right) \\ &= \sum_{m=-\infty}^{\infty} \left\{\cos\left(\frac{2\pi}{P\chi}\frac{[z-mP]^2}{4\pi}\right) - \frac{1}{\sqrt{2}}\right\}\mathrm{sech}\,(\left(\frac{2\pi}{P\chi}\frac{(z-mP)}{2}\right).\end{aligned}$$

The infinite series of translated but otherwise identical copies of a function $G(z-2\pi m)$ is said to be an "imbricate series" and the process of extending $G(z)$ to the infinite series is known variously as the "imbrication of the pattern function $G(x)$, the "periodisation of $G(x)$" or "periodic summation". Good accounts with applications can be found in [5,10, 11].

Section 2 is background on the Poisson Summation Theorem and Imbricate Series. Section 3 is an initiation into the mysteries of Darboux's Principle. The next three sections are novel. Section 4 is a thorough analysis of the Fourier coefficients of the Ramanujan function. This is followed by a short section on oscillations in degree of Fourier coefficients. An important question is: Does Ramanujan's function have a natural boundary? Section 6 makes comparisons with so-called "theta sums" to answer this question in the affirmative.

2. The Poisson Summation Theorem and Imbricate Series

The Poisson Summation Theorem 2.1 shows that every periodic function has a non-Fourier infinite series representation that is the sum of an infinite number of copies of the Fourier Transform of the function $g(n)$ that gives the coefficients of the Fourier series.

Theorem 2.1 (Poisson Summation Theorem)**.** *Any periodic function with period P has the two series representations*

$$f(z)=\chi\sum_{n=-\infty}^{\infty} g(\chi n)\exp(i2\pi nz/P)=\sum_{m=-\infty}^{\infty} G\left(\frac{2\pi}{P\chi}[z-mP]\right),$$

where the series on the left is the usual (complex) Fourier series, the series on the right is the "imbricate" series, χ is an arbitrary positive constant, and $G(k)$ and $g(z)$ are Fourier transforms of one another — i.e.

$$G(k)=\int_{-\infty}^{\infty} g(z)\exp(ikz)\,dz,\quad g(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty} G(k)\exp(-ikz)\,dk.$$

The theorem is valid if the following sufficient conditions are satisfied:

- *(i) $G(k)\in\mathscr{L}^1(\mathbb{R})$, that is, the integral of the absolute value of $G(k)$ over the entire real axis is bounded.*
- *(ii) $g(z)=\mathscr{O}\left((1+|z|)^{-\alpha}\right)$ as $|z|\to\infty$ for some $\alpha>1$.*[†]

For $P=2\pi$ and $\chi=1$,

$$f(z)=\sum_{n=-\infty}^{\infty} g(n)\exp(inz)=\sum_{m=-\infty}^{\infty} G([z-m2\pi]). \tag{2.1}$$

[†]The "Landau gauge symbol" $\mathscr{O}(Q)$ denotes that Q is the order of magnitude in the specified limit. Thus, $\cosh(z)\sim\mathscr{O}(\exp(-z))$ as $z\to\infty$.

Proof. cf. Zayed [35, Page 239]. □

The two series for $\mathfrak{F}(z;\chi)$(2.1) follow by substituting Ramanujan's Fourier Transform pair as $g(n)$ and $G(z)$ in the Poisson Summation formulas. Fig. 1 shows the Fourier coefficient functions $g(x)$ and the pattern function $G(z)$ for an arbitrary but representative value of the parameter χ.

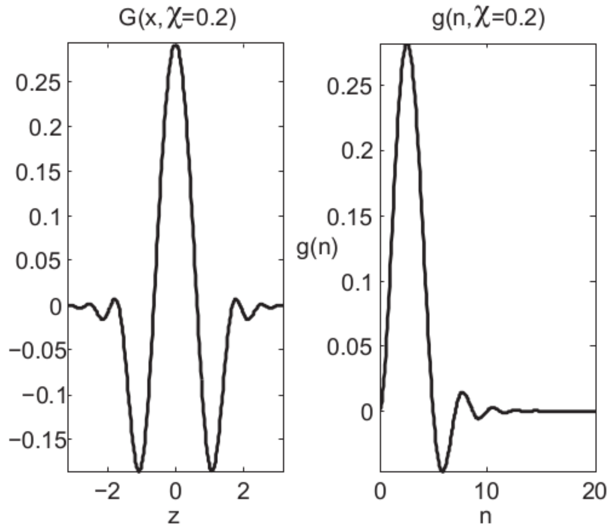

Figure 1: The pattern function $G(z)$ [left] and Fourier coefficient function $g(n)$ [right] for Ramanujan's function for a typical value of the user-choosable parameter χ.

The imbricate series is very useful in applications because it converges rapidly when the Fourier series converges slowly, and slowly only when Fourier convergence is fast [10]. The parameter χ, which multiplies degree in the Fourier coefficients, but divides g the argument in the pattern function, makes this inverse relationship explicit. This inverse relationship between series convergence is really a statement that the width of a function and its Fourier Transform are inversely related; in physics, this is the Heisenberg Uncertainty Principle.

We must leave further general discussion to the references [4, 5, 9, 10, 23, 33]. Before further exploiting the two series for the function derived from Ramanujan's transform pair, we must first review ideas important in Fourier series large degree asymptotics.

3. Darboux's Theorem and Principle

Darboux's Theorem asserts that the coefficients a_n of a power series for a function $f(z)$ are, in the limit $n \to \infty$, controlled by the singularities of $f(z)$ in the complex plane [14–16, 20, 31, 34]. "Singularity" is meant in the usual sense of complex variable theory to denote poles, fractional powers, logarithms and other branch points, and discontinuities of $f(z)$ or any of its derivatives. Each such singularity gives its own additive contribution to the coefficients a_n in the asymptotic limit $n \to \infty$. The contributions decay at exponentially different rates that are determined by the distance of the singularity from the origin. The power series converges within the largest disk centered at $z = 0$ which is singularity free. The singularity/singularities on the circle that bounds the disk of convergence for the power series contribute the slowest-decaying contributions to a_n and therefore asymptotically dominate. The theorem as Darboux published it in 1878 gives the specific contributions to the power series coefficients from poles and branch points. These formulas may be found in the references [14, 15, 20].

Of much greater significance than the power series formulas is that Darboux's Theorem is also a Principle. This asserts that for all types of spectral expansions (Fourier, Chebyshev polynomial, etc., as well as ordinary power series), both the domain of convergence in the complex plane and also the rate of convergence are controlled by the location and strength of the gravest singularity in the complex plane. Again, each singularity makes own contribution to the series coefficient, here the Fourier coefficient $a_n = \chi\, g(n)$, but only those singularities that are nearest to the real axis (for Fourier series) are relevant for sufficiently large n.

A pair of closely related examples is useful in explaining Darboux's Principle. The geometrically converging Fourier series (3.1) has a partial fraction expansion which explicitly shows that this function is singular only through simple poles at $x = \pm ia$ along the imaginary axis and the images of these two poles under the periodicity shift, $x \to x + 2\pi m$ where m is an arbitrary integer:

$$\lambda(x;p) \equiv \frac{(1-p^2)}{(1+p^2) - 2p\cos(x)} = 1 + 2\sum_{n=1}^{\infty} p^n \cos(nx) = 2d \sum_{m=-\infty}^{\infty} \frac{1}{d^2 + (x - 2\pi m)^2}, \quad (3.1)$$

where $d \equiv -\log(p) \to p = \exp(-d)$.

In contrast, the elliptic function Dn[‡] has an *infinite* number of simple poles on the imaginary axis, instead of just two

$$\begin{aligned} \mathrm{Dn}(x;p) &\equiv \frac{1}{2} + 2\sum_{n=1}^{\infty} \left[p^n/(1+p^{2n})\right] \cos(nx) \\ &= B \sum_{m=-\infty}^{\infty} \operatorname{sech}\left[B(x - 2\pi m)\right], \\ B(p) &\equiv \pi/[2\log(1/p)]. \end{aligned}$$

We note that Dn has simple poles at the points

$$x = 2\pi m + id(2j+1), \quad m, j \in \mathbb{Z}.$$

Either by matching poles and residues or by Taylor expanding the $1/(1+p^{2n})$ factor, one can show

$$\mathrm{Dn}(x;p) = \frac{1}{2} + \sum_{j=0}^{\infty} (-1)^j \left\{\lambda(x; p^{2j+1}) - 1\right\}.$$

‡The symbol "dn" is capitalised because we omit an irrelevant constant from the standard definition of "dn"; note also that p is the "elliptic nome", usually symbolised by "q" in the elliptic function literature.

This is a partial fraction expansion, modified so that all poles at a given distance from the real axis are combined into a single term, periodic with real period 2π in x. The poles at a distance of $(2j+1)d$ from the real axis, identical in location and strength to those of $\lambda(x;p^{2j+1})$, contribution an additive term $p^{n[2j+1]}$ to the Fourier coefficient of the elliptic function. Adding together all such contributions gives

$$a_n \sim p^n - p^{3n} + p^{5n} + \cdots .$$

But this is just the power series expansion in powers of p of the exact Fourier coefficients for $\mathrm{Dn}(x;p)$, which are

$$a_n = \frac{p^n}{1+p^{2n}} \sim p^n - p^{3n} + p^{5n} - \cdots$$

as follows from the usual geometric series, $1/(1+p^{2n}) \approx 1 - p^{2n} + p^{4n} - p^{6n} + \cdots$. Fig. 2 is a schematic visualising the poles and their contributions to a_n.

However, the *Fourier coefficients* of the elliptic function are more and more dominated by those of $\lambda(x;p)$ as the degree n of the coefficient increases because

$$\frac{p^n}{1+p^{2n}} = p^n \left\{1 + \mathcal{O}(p^{2n})\right\}, \quad p << 1.$$

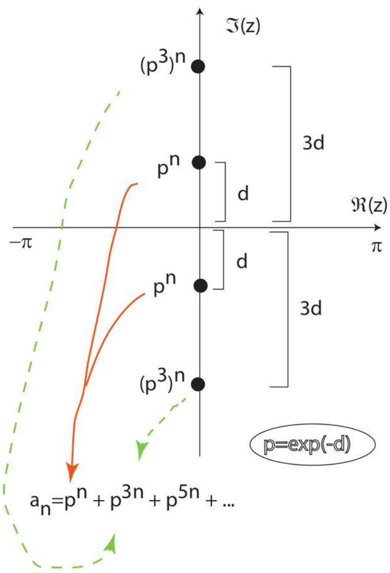

Figure 2: The pattern function (imbrex) for the elliptic function Dn is the hyperbolic secant function, which has an infinite number of poles at distances $\pm id, \pm i3d, \pm i5d, \pm i7d \cdots$ along the imaginary axis. The convergence-limiting poles are the pair at $\pm id$. Each contributes p^n to the Fourier coefficients of the elliptic function where $p = \exp(-d)$ [red arrows]. The singularities at $\pm i3d$ contributes $\exp(-3nd) = \left(p^3\right)^n$. The poles at $z = \pm i5d$ [off the diagram] contribute $(\exp(-5d))^n = p^{5n}$ to the Fourier coefficients. As in Darboux's Theorem for asymptotic power series coefficients, each singularity makes its own contribution to the asymptotic Fourier coefficients.

The singularities closest to the real axis, those of $\lambda(x;p)$ at a distance $a = \log(p)$, completely determine the domain of convergence of the Fourier series of the elliptic function and the leading asymptotic approximation to the Fourier coefficients, $a_n \sim p^n/\{1 + p^{2n}\}$.

This is the typical, generic case.

The pattern function for $\mathfrak{F}(z;\chi)$ is the same as that of the elliptic function except for multiplication by $\{\cos(z^2/(4\pi\chi^2)) - 1/\sqrt{2}\}$.

4. Analysis of the Fourier Coefficients of the Ramanujan Function

4.1. Decay of the Fourier coefficients and the strip of convergence

The period P can be specialised to 2π without lost of generality. The Fourier series and imbricate series are

$$\begin{aligned}\mathfrak{F}(z;\chi) &\equiv 2\chi \sum_{n=1}^{\infty} \sin(\pi\chi^2 n^2)\,\mathrm{sech}\,(\pi\chi n)\cos(nz) \\ &= \sum_{m=-\infty}^{\infty} \left\{\cos\left(\frac{1}{\chi}\frac{[z-m2\pi]^2}{4\pi}\right) - \frac{1}{\sqrt{2}}\right\}\mathrm{sech}\left(\frac{1}{\chi}\frac{(z-m2\pi)}{2}\right).\end{aligned}$$

For large degree n, the coefficient function $g(n)$ simplifies to

$$\sin(\pi\chi^2 n^2)\frac{1}{2}\exp(-\pi\chi n)\{1 + \mathcal{O}(\exp(-2\pi\chi\, n))\}.$$

Note the Fourier cosine coefficient is $a_n = \chi\, g(n)$.

The exponential decay of the Fourier coefficients, proportional to an exponential with an argument linear with degree, is "geometric convergence" in the usual jargon of Fourier and Chebyshev polynomial spectral methods [7]. This is generic for the Fourier coefficients of a function that is periodic and analytic on the real axis. Indeed, it is generic for power series, including the geometric series $1/(1-x) = 1 + x + x^2 + \cdots + x^N + x^{N+1}/(1-x)$, at any point inside its radius of convergence. For the other typical functions, the exponential may be modulated by a power of n or a logarithm of degree, but asymptotically these are overwhelmed by exponential. Classic theory [7] asserts that the Fourier series converges in the widest strip centered on the real axis, which is free of singularities of $f(z)$. Is this true here? And where are the singularities of the function $\mathfrak{F}(z,\chi)$ introduced here?

We shall address singularity location first. The singularities of the series are the poles and branch points, etc., of the pattern function. Here, the pattern function has a factor $\mathrm{sech}\,(z/(2\chi))$ which has an infinite set of poles on the imaginary axis. The poles closest to the real axis are located where the argument of the hyperbolic secant, is $\pm\pi/2i$, that is, $z_s = i\pi\chi$. If these are indeed the poles of the pattern function, then the predicted convergence strip is thus $|\Im(z)| \le \pi\chi$. (Recall that $|\exp(iz)| = \exp(n|\Im(z)|)$. At the boundary of the convergence strip, the exponential decay of the coefficients proportional to $\exp(-n\chi\pi)$ is exactly canceled by the growth of $\exp(inz)$ with increasing $\Im(z)$.) Thus, the *decay* of the coefficients of the Ramanujan function and the convergence domain of its Fourier series seem to be boringly typical.

However, this is only an illusion because of the following.

Theorem 4.1. *The pattern function for the Ramanujan function is the product of*

$$\Phi(z;\chi) \equiv \cos\left(\frac{z^2}{4\pi\chi^2}\right) - \frac{1}{\sqrt{2}}$$

with

$$\Psi(z;\chi) \equiv \operatorname{sech}\left(\frac{z}{2\chi}\right).$$

Then

1. $\Psi(z;\chi)$ *has simple poles at*

$$z_s(m) = \pm i(2m-1)\pi\chi, \quad m = 1, 2, \cdots.$$

2. $\Phi(z;\chi)$ *has simple zeros at all of these same points.*
3. *The Ramanujan pattern function* $\Phi(z;\chi)\Psi(z;\chi)$ *is an entire function with no singularities in the complex* z*-plane except at infinity.*

Proof. As noted above, the function $\operatorname{sech}(z/(2\chi))$ has simple poles at the points $z_s(m)$ as follows trivially from the definition $\operatorname{sech}(z) = 1/\cosh(z) = 1/\cos(iz)$ and the known zeros of the cosine function. The other factor of the pattern function is

$$\Phi \equiv \cos\left(\frac{z^2}{4\pi\chi^2}\right) - \frac{1}{\sqrt{2}}.$$

At the singularities of the hyperbolic secant function,

$$\begin{aligned}\Phi(z_s)(m) &\equiv \cos\left(\frac{z_s(m)^2}{4\pi\chi^2}\right) - \frac{1}{\sqrt{2}} = \cos\left(\frac{i^2(2m-1)^2\pi^2\chi^2}{4\pi\chi^2}\right) - \frac{1}{\sqrt{2}} \\ &= \cos\left(\left(m - \frac{1}{2}\right)^2 \pi\right) - \frac{1}{\sqrt{2}} = \cos\left(\left[m^2 - m + 1/4\right]\pi\right) - \frac{1}{\sqrt{2}} \\ &= \cos\left(\frac{\pi}{4}\right) - \frac{1}{\sqrt{2}} = 0,\end{aligned}$$

where we have used the fact that $m^2 - m = m(m-1)$ is even for any integer m — note that one of the pair $(m, m-1)$ is always even — and also that $\cos(2\pi n + q) = \cos(q)$ for any q and any integer n. The third proposition follows from l' Hôpital's Rule. □

When the pattern function is entire, this usually means that imbrication of the pattern function is entire and its Fourier series will converge throughout the entire complex plane except at infinity. Indeed, before completing this investigation, the author believed this was always true. In this theorem, we show that the derived-from-Ramanujan function is a *counterexample*.

It is fortunately the case that given the explicit form of the Fourier coefficients, it is possible to determine the convergence region of the Fourier series without employing any information about its singularities.

Theorem 4.2 (Strip of Convergence). *For the Ramanujan function $\mathfrak{F}(z;\chi)$, the Fourier series converges within the infinite rectangle that includes the entire real axis and is bounded by*

$$|\Im(z)| < \pi\chi.$$

The domains of convergence and divergence for the imbricate series of the Ramanujan function are identical to those of the Fourier expansion. We conjecture that the Fourier series diverges everywhere outside this strip but lack a rigorous proof.

Proof. For large degree, the Fourier coefficients asymptote to

$$a_n = 2\chi g(n) = 2\chi \sin(\pi\chi^2 n^2)\,\mathrm{sech}\,(\pi\chi n) \sim \sin(\pi n^2)\frac{1}{2}\exp(-\pi\chi n).$$

The Fourier basis function, $\exp(inz)$, is bounded tightly in absolute value by $\exp(n|\Im(z)|)$. It follows that the n-th term in the series, $2g(n)\cos(nz)$, is bounded by

$$\frac{1}{2}\exp\{(|\Im(z)| - \pi\chi)n\}.$$

The n-th term thus decreases exponentially fast as $n \to \infty$, provided that $|\Im(z)| < \pi\chi$. This proves that Fourier series converges within the strip claimed by the theorem.

Although we lack a rigorous proof of divergence outside the strip of provable convergence, note that either $\exp(inz)$ or $\exp(-inz)$ is blowing up exponentially at a faster rate for $|\Im(z)| > \pi\chi$ than the rate at which the $\mathrm{sech}\,(\pi\chi n)$ is decaying with n. This falls short of a proof because of the very rapid oscillations of the coefficients due to $\sin(\pi\chi^2 n^2)$ factor; we have not yet been clever enough to bound these cancellations except experimentally.

Next, we analyse convergence for the imbricate series. Let $z = \chi(x + iy)$. The pattern function is

$$\begin{aligned}
G &= \Phi\Psi,\\
\Phi &= \frac{1}{2}\left\{\exp\left(\frac{i(x^2-y^2)}{4\pi}\right)\exp\left(\frac{y}{2\pi}x\right) + \exp\left(\frac{-i(x^2-y^2)}{4\pi}\right)\exp\left(-\frac{y}{2\pi}x\right)\right\} - 1/\sqrt{2}\\
&\sim \exp\left(\frac{\mathrm{sign}(x)i(x^2-y^2}{4\pi}\right)\exp\left(\frac{y}{2\pi}|x|\right),\\
\Psi &= \frac{2}{\exp(x/2)\exp(iy/2) + \exp(-x/2)\exp(-iy/2)} \sim 2\exp\left(\frac{-|x|}{2}\right)\exp\left(-\mathrm{sign}(x)i\frac{y}{2}\right).
\end{aligned}$$

It follows that

$$G \sim \exp\left(\frac{\mathrm{sign}(x)i(x^2-y^2)}{4\pi}\right)\exp\left(-\mathrm{sign}(x)i\frac{y}{2}\right)\exp\left\{\left(\frac{y}{2\pi} - \frac{1}{2}\right)|x|\right\}.$$

Thus, the pattern function decays with $|\Re(z)|$ if $|\Im(z)| < \pi\chi$ but grows exponentially otherwise. The imbricate series converges only if the pattern function is decaying. ☐

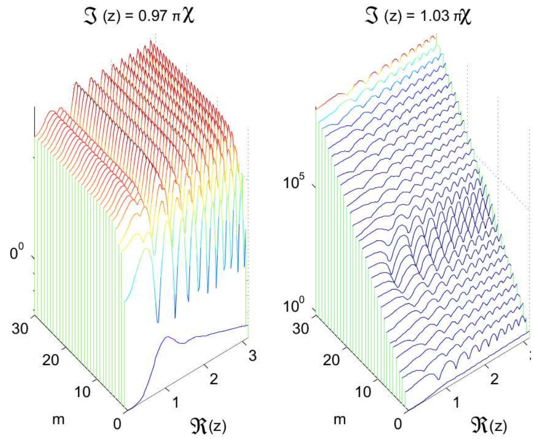

Figure 3: Imbricate partial sum Imb_m for fixed $\Im(z)$ and varying $\Re(z)$ slightly inside the strip of convergence (left) and slightly outside the strip of convergence (right). Absolute values are plotted using a logarithmic vertical scale.

Fig. 3 is a visualisation of the convergence and divergence of imbricate series. Define the lowest partial sum of the imbricate series to be the pattern function and define higher partial sums to be

$$\mathrm{Imb}_m = \sum_{j=-m}^{m} G(z - 2\pi j).$$

Inside the convergence strip, if barely, the first four or partial sums differ from one another, but the remaining twenty-five are identical because the imbricate series has converged to within graphical limits. Sightly beyond the boundary of the convergence strip, the partial sums grow exponentially, and the waterfall plot has a steep slope.

5. Oscillations in Degree

What first caught the author's eye about Ramanujan's Fourier Transform pair is that the periodic function $\mathfrak{F}(z;\chi)$ constructed from this pair would have a wide discrepancy between the rate at which the Fourier coefficients *decay* with degree and the rate at which the same coefficients *oscillate*. The decay is an exponential with an argument *linear* in degree n, but the oscillations are a sine function with an argument *quadratic* in degree.

It is not at all unusual for Fourier coefficients to oscillate. Indeed, if $f(z)$ has a branch point singularity of the form z^{σ}, located along the imaginary axis, its asymptotic ($n \gg 1$) coefficients will decrease monotonically without changes of sign. However, elementary trigonometric identities show that $f(z-s)$ for a shift s will have the cosine coefficients $a_n(s)$ and sine coefficients $b_n(s)$

$$a_n(s) = a_n^{unshifted}\cos(ns) - b_k^{unshifted}\sin(ns),$$
$$b_n(s) = a_n^{unshifted}\sin(ns) + b_n^{unshifted}\cos(ns).$$

These translation-induced oscillations are trigonometric functions of *linear* argument. It is not possible for translation along the real axis to eliminate the oscillations of the Fourier coefficients of the Ramanujan function.

To be sure, examples have been long known where the Fourier coefficients oscillate at a rate which is a higher power of degree n than the exponential decay. For example, if $f(z)$ is C^{∞} but not analytical everywhere on the interval $z \in \varepsilon$, then its Fourier coefficients will converge at a subgeometric rate. In particular, Miller [22] showed that the Fourier coefficients of $\exp(-1/\cos(z))$ have root-exponential convergence, falling as $\exp(-p\sqrt{n})$. The coefficients of the translated function $\exp(-1/\cos(z-s))$ will oscillate like $\cos(ns)$ and $\sin(ns)$.

When the oscillations are controlled by trigonometric functions with an argument linear in degree, the oscillations have fixed period in degree. When the phase is quadratic in n, as for the Ramanujan function, the wavelength is shortening without bound.

This is very bad for sum acceleration methods that extract accurate answers from slowly-converging series. Exponential convergence vanishes at $\Im(z) = \pi\chi$, whose neighbourhood is therefore prime territory for such accelerations, but deeper knowledge of the series is required by most of the standard accelerators. Also, the nature of the convergence-limiting singularities along this line are so far unresolved. These observations furnish at least two motives for the analysis of the Fourier series along this line in the next section.

6. Theta Sums: Life on the Boundaries of the Convergence Strip

Theorem 6.1. *On the boundary of the strip of convergence, $y = \Im(z) = \pi\chi$, the partial sums of the real part of the Fourier series for $\mathfrak{F}(x + i\pi\chi; \chi)$, are the sums of four Gaussian sums,*

$$\begin{aligned}\Re(\mathfrak{F}_N(x + i\pi\chi; \chi)) &= 2\chi \sum_{n=1}^{N} \sin(\pi\chi^2 n^2)\cos(nx) \\ &= 2\chi\,\{S(N,\tau,ix) + S(N,\tau,-ix) + S(N,-\tau,ix) + S(N,-\tau,-ix)\},\end{aligned} \tag{6.1}$$

where

$$S(N,\tau,ix) = 2\sum_{n=1}^{N} \exp(i\tau n^2)\exp(nx)$$

with $i\tau = \pi\chi^2$.

Proof. Without approximation.

$$\begin{aligned}\mathfrak{F}(z;\chi) &\equiv \chi \sum_{n=-\infty}^{\infty} \sin(\pi\chi^2 n^2)\,\mathrm{sech}\,(\pi\chi n)\exp(inz) \\ &= 2\chi \sum_{n=1}^{\infty} \sin(\pi\chi^2 n^2)\frac{\cos(nz)}{\cosh(\pi\chi n)} \\ &= 2\chi \sum_{n=1}^{\infty} \sin(\pi\chi^2 n^2)\frac{\cos(nx)\cosh(ny) - i\sin(nx)\sinh(ny)}{\cosh(\pi\chi n)},\end{aligned}$$

where we have applied $\cos(x+iy)=\cos(x)\cosh(y)-i\sin(x)\sinh(y)$.

Let $y=\pi\chi$ and let $\digamma$ denote $\mathfrak{F}$ along this line parallel to the real axis. Taking the real part of $\digamma$ gives the first line of (6.1). Writing both the sine and cosine as complex exponentials turns each term into four. The rest is merely definitions and notation. □

The four sums have similar properties, so it suffices to discuss just

$$S(N,\tau,ix)=\sum_{n=1}^{N}\exp(i\tau n^2)\exp(inx).$$

The series converges only for $\Im(\tau)>0$, which is why a truncation N is usually introduced as here to understand what happens when τ is real. The sum is also, in the limit $N\to\infty$, a Gaussian theta function. As such, it is known that the limit $\Im(\tau)=0$ is a natural boundary. It is not possible to analytically continue a function past its natural boundary because the function is singular everywhere along the natural boundary rather than at isolated points.

Sir Michael Berry [2] analyses a function of similar form in:

$$K(z,\tau)\equiv\sum_{n=-\infty}^{\infty}\exp\left(-i\pi\tau\left(n+\frac{1}{2}\right)^2\right)\exp(i(2n+1)z).$$

He writes that “this is a theta function (Gauss sum) on its natural boundary. For rational τ, the [indefinite] integral is piecewise constant and describes fractional quantum levels and the fractional Talbot effect [in optics]. For irrational τ, the graph of the wave intensity as a function of z is a fractal with dimension 3/2. As a function of τ, the graph has dimension 7/4."

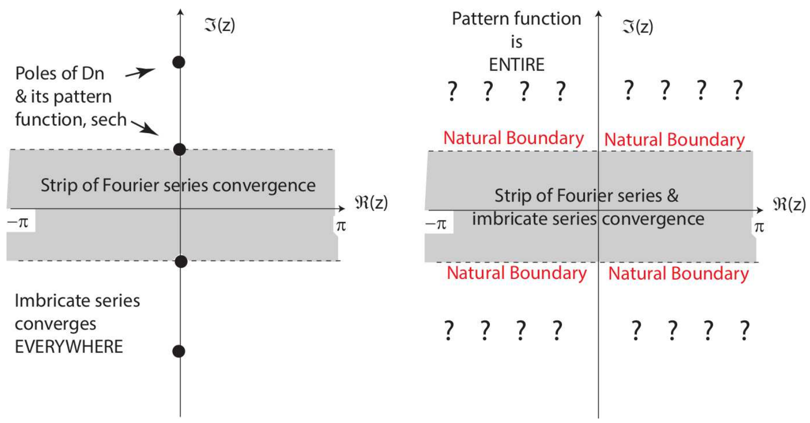

Figure 4: Left: the elliptic function Dn and its pattern function, sech, share an infinite number of evenly spaced simple poles along the imaginary axis, marked by the black disks [and continuing indefinitely beyond the limits of the graph]. Right: the function $\mathfrak{F}$ has a pattern function or imbrex with no singularities, so the black disks are absent. The strip of convergence of the Fourier series for both is the largest strip centered on the real axis which is singularity-free. For the Ramanujan function, the boundary of the strip is a natural boundary. The question marks reflect the uncertainty that arises because a function cannot be analytically continued beyond its natural boundaries.

There is a fascinating theory of Gauss sums, theta functions and limits of Fourier partial sums, but this is beyond the scope of this article.

Fig. 4 is a visual summary of the differences and similarities between a generic function like the elliptic function Dn and the function derived from Ramanujan, $\mathfrak{F}$.

7. Conclusions and Open Questions

Despite many decades of work, there are still many lacunae in the theory of Fourier series. Ramanujan's integrals have enabled us to generate a periodic function whose pattern function is explicit. This in turn allows us to probe deeper into some of these gaps.

The first conclusion is that the interpretation and application of Darboux's Principle to periodic functions and Fourier series is more complicated than hitherto suspected. The simplest interpretation for Fourier series is that the isolated singularities of the pattern function $G(z)$ determine the width of the strip of Fourier convergence. $\mathfrak{F}(z,\chi)$ is a counterexample. The function $G(z)$ is entire, but the width of the strip of convergence is not infinite. However it remains true that the Fourier series converges within the largest strip centered on the real axis which is free of singularities. It also remains true that the singularities of $f(x)$ determine the rate of convergence on the real axis and also the width of the convergence strip.

The second conclusion is that $\mathfrak{F}(z;\chi)$ has natural boundaries on the lines $\Im(z) = \pm\chi\pi$, which are also the boundaries of the strip of convergence of both the Fourier and imbricate series. Functions with natural boundaries have been known for many decades, but most of the literature is focused on lacunary series. These are series in which most of the Fourier coefficients are zero with steadily increasing gaps ("lacunae") in degree between adjacent nonzero coefficients such as

$$f_{lac}(z) \equiv \sum_{n=0}^{\infty} 2^{-n}\cos(3^n z) = 1 + \frac{1}{2}\cos(3z) + \frac{1}{4}\cos(9z) + \frac{1}{8}\cos(27z) + \frac{1}{16}\cos(81z) + \cdots,$$

which is a specialisation of the Weierstrass-Mandelbrot function whose fractal curves and limits are discussed by Berry and Lewis [3]. The function derived from Ramanujan's transforms has an infinite series with a geometric rate of convergence on the real axis and no gaps in degree — indeed all possible cosine coefficients are nonzero, a sort of "anti-lacunary" function in that regard, and has a natural boundary anyway.

The third conclusion is that it is possible for Fourier coefficients to decay geometrically as $\exp(-\mu z)$ for some positive μ and yet have oscillations proportional to $\sin(\omega n^2)$ for some real constant ω (here equal to $\pi\chi^2$) which is a *quadratic* function of degree.

Many open questions remain to keep future scholars busy.

1. Does the derived-from-Ramanujan function $\mathfrak{F}(z;\chi)$ satisfy any simple differential or integral equations?

2. Can it be computed by an iteration or some method that bypasses its Fourier and imbricate series?

3. Are there any interesting properties of its dependence on the parameter χ?

4. Does it arise in applications?

5. Are there larger identifiable classes of periodic functions for which Fourier convergence is limited by natural boundaries?

The reader can doubtless think of many more.

More than most contemporary mathematicians, Ramanujan valued specific examples. Metaphorically, the abstract formalism of modern mathematics is the mindset of a theoretical physicist, searching for a Grand Unified Theory of Everything. Ramanujan had also the love of mathematical botany, the collection of interesting individuals. Derived from one of his examples, the function $\mathfrak{F}(z;\chi)$ is a specimen that is interesting in itself but which also opens a door to wider questions.

Acknowledgments

This was supported by the National Science Foundation of the US under DMS-1521158.

References

[1] B.C. Berndt, *Ramanujan's Notebooks, Part I*, Springer (1985).
[2] M.V. Berry, *Quantum and optical arithmetic and fractals*, in: *The Mathematical Beauty of Physics*, J.M. Drouffe and J.B. Zuber (Eds), pp. 281–294, World Scientific Press (1997).
[3] M.V. Berry and Z.V. Lewis, *On the Weierstrass-Mandelbrot fractal function*, Proc. R. Soc. London A **370**, 459–484 (1980).
[4] J.P. Boyd, *Theta functions, Gaussian series, and spatially periodic solutions of the Korteweg-de Vries equation*, J. Math. Phys. **23**, 375–387 (1982).
[5] J.P. Boyd, *New directions in solitons and nonlinear periodic waves: Polycnoidal waves, imbricated solitons, weakly non-local solitary waves and numerical boundary value algorithms*, in: *Advances in Applied Mechanics* ***27***, T.Y. Wu and J.W. Hutchinson (Eds), pp. 1–82, Academic Press (1989).
[6] J.P. Boyd *Weakly Nonlocal Solitary Waves and Beyond- All-Orders Asymptotics: Generalized Solitons and Hyperasymptotic Perturbation Theory*, Mathematics and Its Applications **442**, Kluwer (1998).
[7] J.P. Boyd, *Chebyshev and Fourier Spectral Methods*, Dover (2001).
[8] J.P. Boyd, *Solving Transcendental Equations: The Chebyshev Polynomial Proxy and Other Numerical Rootfinders, Perturbation Series and Oracles*, SIAM (2014).
[9] J.P. Boyd, *Dynamics of the Equatorial Ocean*, Springer-Verlag (2018).
[10] J.P. Boyd *Imbricate-Fourier Series: Applications of the Poisson Summation to Solitary Waves, Special Functions, Crystal Lattice Sums and Other Physics and Engineering Problems*, SIAM (To appear).
[11] J.P. Boyd and S. E. Haupt, *Polycnoidal waves: Spatially periodic generalizations of multiple solitary waves*, in: *Nonlinear Topics of Ocean Physics: Fermi Summer School, Course LIX*, A.R. Osborne (Ed), pp. 827–856, North-Holland (1991).
[12] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, *Spectral Methods: Fundamentals in Single Domain*, Springer-Verlag (2006).

[13] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, *Spectral Methods: Evolution to Complex Geometries and Application to Fluid Dynamics*, Springer-Verlag (2007).

[14] M.G. Darboux, *Mémoire sur l'approximation des fonctions de très-grands nombres, et sur une classe étendue de développements en série*, Journal of Mathematiques Pures Appliques **4**, 5–56 (1878).

[15] M.G. Darboux, *Mémoire sur l'approximation des fonctions de très-grands nombres, et sur une classe étendue de développements en série deux*, Journal of Mathematiques Pures Appliques **4**, 377–416 (1878).

[16] R.B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation*, Academic (1973).

[17] B.A. Finlayson, *The Method of Weighted Residuals and Variational Principles*, SIAM (2013).

[18] D. Funaro, *Polynomial Approximation of Differential Equations*, Springer-Verlag (1992).

[19] J.S. Hesthaven, S. Gottlieb and D. Gottlieb, *Spectral Methods for Time-Dependent Problems*, Cambridge University Press (2007).

[20] C. Hunter and B. Guerrieri, *Deducing the properties of singularities of functions from their Taylor series coefficients*, SIAM J. Appl. Math. **39**, 248–263 (1980).

[21] R. Kanigel, *The Man Who Knew Infinity: A Life of the Genius Ramanujan*, Washington Square Press (1992).

[22] G.F. Miller, *On the convergence of Chebyshev series for functions possessing a singularity in the range of representation*, SIAM J. Numer. Anal. **3**, 390–409 (1966).

[23] A. Osborne, *Nonlinear Ocean Waves and the Inverse Scattering Transform*, International Geophysics **67**, Academic (2010).

[24] R. Peyret, *Spectral Methods for Incompressible Viscous Flow*, Applied Mathematical Sciences **148**, Springer-Verlag (2002).

[25] S. Ramanujan, *Some definite integrals connected with Gauss's sums*, Messenger of Mathematics **44**, 75–85 (1915).

[26] S. Ramanujan, *Some definite integrals*, J. Indian Math. Soc. **11**, 81–87 (1919).

[27] J. Shen, T. Tang and L.L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*, Springer Series in Computational Mathematics **41**, Springer (2013).

[28] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford University Press (1948).

[29] L.N. Trefethen, *Spectral Methods in Matlab*, SIAM (2000).

[30] L.N. Trefethen, *Approximation Theory and Approximation Practice*, SIAM (2012).

[31] H. Volkmer, *On some conjectures of Askey*, Asymptotic Anal. **27**, 321–334 (2001).

[32] G.N. Watson, *Ramanujan's integrals and Gauss's sums*, Quart. J. Math. **7**, 175–180 (1936).

[33] G.B. Whitham, *Comments on periodic waves and solitons*, IMA J. Appl. Math. **32**, 353–366 (1984).

[34] R. Wong, *Asymptotic Approximation of Integrals*, Academic Press, (1989).

[35] A.I. Zayed, *Handbook of Function and Generalized Function Transformations*, Mathematical Sciences Reference Series **3**, CRC Press (1996).