Dynamic Output Feedback Control of Discrete-Time Nonlinear Quadratic Systems with Stochastic Parametric Uncertainty and Missing Measurements

Yujing Shi\(^1\), Shanqiang Li\(^1,2\) and Yueru Li\(^1\)

\(^1\)Department of Applied Mathematics, Harbin University of Science and Technology, Harbin 150080, PR. China.

\(^2\)College of Automation, Harbin Engineering University, Harbin 150001, PR. China.

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Abstract. Finite-time dynamic output feedback control for a class of discrete-time nonlinear quadratic systems with stochastic parametric uncertainty, exogenous disturbance and missing measurements, modeled by a Bernoulli distributed stochastic variable, is considered and sufficient conditions for FTSB under a dynamic output feedback controller are provided. As a consequence, a sufficient condition for FTSS is derived. A numerical example demonstrates the effectiveness of the method.

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1. Introduction

In recent years, the Lyapunov asymptotic stability of dynamical systems on infinite time intervals has been energetically studied. Nevertheless, in practical applications, it is more important to establish the system trajectories over fixed finite time intervals [12, 16]. For example, this problem is connected to saturation elements in closed-loop systems and to controlling the trajectories of space vehicles during prescribed time intervals. The concept of finite-time stability (FTS) was introduced by Kamenkov [10] in 1953. Since then, substantial efforts have been spent on FTS analysis [8, 9, 21]. Later on the finite-time boundedness (FTB) of dynamical systems with exogenous disturbances has been considered [4, 5] and, recently, FTS and FTB of deterministic systems have been extended to stochastic ones. In addition, the problems of finite-time stochastic boundedness (FTSB) and finite-time stochastic stabilisation (FTSS) for Markovian jump systems with partially unknown transition probabilities are discussed in [22, 23].
During the past years, the control problems for nonlinear systems have attracted a wide attention because of numerous practical applications. In particular, quadratic systems are often encountered in chemical reactors, electrical power systems \cite{15}, robotics \cite{11} and biology \cite{13}. More specifically, the domain of attraction (DA) problems are discussed in \cite{3,7}, finite-time stability and stabilisation problems arising in the design of optimal control strategies is considered in \cite{20}, a guaranteed cost control in \cite{1,2}, the sufficient conditions for finite-time boundedness and stabilisation of continuous-time quadratic systems are derived in \cite{6}.

Another group of actively studied non-linear systems includes systems with missing measurements \cite{14,18}. Missing measurement phenomena can be modeled by binary switching sequences. A binary switching sequence is specified by a conditional probability distribution and enters into the system observation. It can be considered as a Bernoulli distributed white sequence taking values 0 and 1 — cf. Refs. \cite{17,19}. Nevertheless, it is often very difficult, if not impossible, to accurately determine certain system parameters. Therefore, for discrete-time nonlinear quadratic systems with stochastic parametric uncertainty, it is very important to consider the stochastic stabilisation. However, to the best of author’s knowledge, so far the FTSB and FTSS problems for such kind systems have not been studied. We also note that the most works on control problems deal only with state feedback control strategies, although in many engineering systems the state is not always accessible and the output feedback control is of practical significance. In this work, we consider a dynamic output feedback controller for discrete-time nonlinear quadratic systems with stochastic parametric uncertainties and missing measurements. In particular, we establish sufficient conditions to guarantee the FTSB and FTSS for closed-loop systems.

The paper is organised as follows. In Section 2, the target plant is described by a class of discrete-time nonlinear quadratic systems with stochastic parametric uncertainty and exogenous disturbance. Then the measurement-missing phenomena are proposed and a dynamic output feedback controller is developed. In Section 3, a sufficient condition of FTSB for the closed-loop system is presented. It yields a sufficient condition of FTSS for closed-loop system without exogenous disturbance. An example in Section 4 demonstrates the efficiency of the result obtained.

\section{2. Preliminaries and Problem Formulation}

We consider the discrete-time nonlinear quadratic system with stochastic parametric uncertainty and exogenous disturbance

\begin{equation}
\begin{aligned}
x(k + 1) &= (A + \alpha(k)\Delta A)x(k) + F(x(k)) + Bu(k) + B_w w(k), \\
y(k) &= C x(k) + D_w w(k), \\
w(k + 1) &= G w(k),
\end{aligned}
\end{equation}

where \(x(k) \in R^n\) is the state vector; \(u(k) \in R^m\) the control input, \(w(k) \in R^p\) the exogenous disturbance, \(y(k) \in R^r\) the output, \(A \in R^{n \times n}\), \(B \in R^{n \times m}\), \(B_w \in R^{n \times p}\), \(C \in R^{r \times n}\), \(D_w \in R^{r \times p}\), \(G \in R^{p \times p}\) are constant matrices and
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\[ F(x(k)) := \begin{bmatrix} x^T F_1 \\ \vdots \\ x^T F_n \end{bmatrix} \quad x = F_k(x)x \]

with \( F_i \in \mathbb{R}^{n \times n}, i = 1, 2, \cdots, n \).

The matrix \( \Delta A \in \mathbb{R}^{n \times n} \) represents a norm-bounded parameter uncertainty,

\[ \Delta A = DF(k)E, \]

where \( D \) and \( E \) are known constant matrices and \( F(k) \) is an unknown matrix function such that

\[ F^T(k)F(k) \leq I. \]

Stochastic variable \( \alpha(k) \) is a Bernoulli distributed white sequence with a probability distribution

\[ \text{prob}\{\alpha(k) = 1\} = \alpha, \quad \text{prob}\{\alpha(k) = 0\} = 1 - \alpha. \]

In practical applications, the output \( y(k) \) may not be always available because of missing measurements. To model the missing-measurements, we use the following equation:

\[ \tilde{y}(k) = \theta(k)Cx(x) + D_ww(k), \quad (2.2) \]

where \( \tilde{y}(k) \) represents the actually available signal for the plant and \( \theta(k) \) is a Bernoulli distributed stochastic variable, not related to \( \alpha(k) \) and such that

\[ \text{prob}\{\theta(k) = 1\} = \theta, \quad \text{prob}\{\theta(k) = 0\} = 1 - \theta. \]

We are looking for a dynamic output feedback controller for system (2.1) of the form

\[ \dot{x}(k+1) = A_c\dot{x}(k) + B_c\tilde{y}(k), \quad u(k) = C_c\dot{x}(k), \quad (2.3) \]

where \( \dot{x}(k) \in \mathbb{R}^n \) is the controller state and \( A_c \in \mathbb{R}^{n \times n}, B_c \in \mathbb{R}^{n \times r}, C_c \in \mathbb{R}^{m \times n} \) are the gain matrices to be determined.

Setting \( \eta(k) = [x^T(k) \dot{x}^T(k)]^T \) and using (2.1)-(2.3), we write

\[ \eta(k+1) = \begin{bmatrix} A + (\alpha(k) - \alpha)\Delta A + (\theta(k) - \theta)\bar{A} \end{bmatrix} \eta(k) + \bar{B}w(k), \quad (2.4) \]

where

\[ \bar{A} = \begin{bmatrix} A + \alpha \Delta A + F_k(x) & BC_c \\ \theta B_c C & A_c \end{bmatrix}, \quad \Delta \bar{A} = \begin{bmatrix} \Delta A & 0 \\ 0 & 0 \end{bmatrix}, \]

\[ \bar{A} = \begin{bmatrix} 0 & 0 \\ B_c C & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_w \\ B_c D_w \end{bmatrix}. \]
In space $R^n$ we consider the polytopes

\[
\Omega_0 = \text{conv}\{\eta_1^{(1)}, \eta_2^{(2)}, \ldots, \eta_p^{(p)}\} = \{\eta \in R^n : v_0^T \eta \leq 1, k = 1, 2, \ldots, q\},
\]

\[
\Omega = \text{conv}\{\eta_1^{(1)}, \eta_2^{(2)}, \ldots, \eta_p^{(p)}\} = \{\eta \in R^n : v_k^T \eta \leq 1, k = 1, 2, \ldots, q\},
\]

where $v_{0k}, v_k \in R^n, p$ and $q$ are suitable integers, $\eta_{i0}^{(i)}$ is the $i$-th vertex of $\Omega_0$, $\eta_i^{(i)}$ is the $i$-th vertex of $\Omega$, and $\text{conv}\{\cdot\}$ is the convex hull of the corresponding set.

For example, the box

\[
\Omega = [-1,2] \times [-1,3]
\]

considered in $[1,2]$, can be written in form (2.5) with

\[
\eta^{(1)} = \begin{bmatrix} 2 & -1 \end{bmatrix}^T, \quad \eta^{(2)} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T, \quad \eta^{(3)} = \begin{bmatrix} -1 & 3 \end{bmatrix}^T, \quad \eta^{(4)} = \begin{bmatrix} -1 & -1 \end{bmatrix}^T
\]

and

\[
v_1^T = \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix}, \quad v_2^T = \begin{bmatrix} -1 & 0 \end{bmatrix}, \quad v_3^T = \begin{bmatrix} 0 & \frac{1}{3} \end{bmatrix}, \quad v_4^T = \begin{bmatrix} 0 & -1 \end{bmatrix}.
\]

In what follows we need the following definitions and lemmas.

**Definition 2.1.** Let $\Omega_0, \Omega$ be polytopes and $M$ a positive integer. The closed-loop system (2.4) with $w(k) = 0$ is called finite-time stochastic stable with respect to $(\Omega_0, \Omega, M)$ if for any $k \in \{1,2,\cdots,M\}$ the condition $\eta(0) \in \Omega_0$ yields $\eta(k) \in \Omega$.

**Definition 2.2.** Let $\Omega_0, \Omega$ be polytopes, $M$ be a positive integer and $d > 0$. The closed-loop system (2.4) is called finite-time stochastic bounded with respect to $(\Omega_0, \Omega, M, d)$ if for any $k \in \{1,2,\cdots,M\}$, the conditions $\eta(0) \in \Omega_0$ and $w^T(0)w(0) \leq d$ yield $\eta(k) \in \Omega$.

In this paper, we always assume that $\Omega_0$ and $\Omega$ have the same number of vertices.

**Remark 2.1.** The FTSS and FTSB mean that the state trajectory starting within polytope $\Omega_0$ does not leave the polytope $\Omega$ within a finite time. Moreover, if there is a controller such that the closed-loop system (2.4) is FTSB, then it can lead to FTSS. The converse is not true.

**Lemma 2.1.** For any symmetric matrix

\[
S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}
\]

the following assertions are equivalent:

1. $S < 0$.
2. $S_{11} < 0$ and $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$.
3. $S_{22} < 0$ and $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$. 
Lemma 2.2. If $W$, $S$ and $P > 0$ are real matrices with compatible dimensions, then

$$WS + (WS)^T \leq WPW^T + S^TP^{-1}S.$$ 

Lemma 2.3. If $Q = Q^T$, $D$, $E$ are real matrices of appropriate dimensions and $F^TF \leq I$, then the inequality

$$Q + DFE + (DFE)^T < 0$$

holds if and only if there is an $\epsilon > 0$ such that

$$\begin{bmatrix} Q & \epsilon D & E^T \\ \epsilon D^T & -\epsilon I & 0 \\ E & 0 & -\epsilon I \end{bmatrix} < 0.$$ 

3. The Main Results

The goal of this work is to develop a dynamic output feedback controller of the form (2.3) such that the resulting closed-loop system of (2.4) is FTSB and FTSS.

Theorem 3.1. Let $\Omega_0$, $\Omega$ be polytopes (2.5), $M$ a positive integer, $d > 0$ and $\beta \geq 1$. If there are matrices $P > 0$, $Q > 0$, $X > 0$, $Y > 0$, $A_s, B_s, C_s$ such that

$$\begin{bmatrix} 1 & \nu_k^T \\ * & P \end{bmatrix} \geq 0, \quad k = 1, 2, \cdots, q,$$  

$$\eta^{(i)}_{\Omega_0} P \eta^{(i)}_{\Omega_0} \leq 1, \quad i = 1, 2, \cdots, p,$$  

$$\begin{bmatrix} \tilde{\Theta} & \epsilon H & Z^T \\ * & -\epsilon I & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0,$$  

$$\beta^M \lambda_{\max}(P) \max_i \|\eta^{(i)}_{\Omega_0}\|^2 + \beta^M \lambda_{\max}(Q) d \leq 1,$$  

where

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Psi}_1 & \tilde{\mu} & \theta \\ * & -I & 0 \\ * & * & -I \end{bmatrix}, \quad \tilde{\Psi}_1 = \begin{bmatrix} \beta \pi_1 & 0 & \pi_2 & 0 & \sqrt{\theta(1-\theta)} \pi_3 & 0 \\ * & -\beta Q & \pi_4 & 0 & 0 & G^T Q \\ * & * & \pi_1 & 0 & 0 & 0 \\ * & * & * & \pi_1 & 0 & 0 \\ * & * & * & * & \pi_1 & 0 \\ * & * & * & * & * & -Q \end{bmatrix}.$$  

$$\pi_1 = \begin{bmatrix} -Y & -I \\ * & -X \end{bmatrix}, \quad \pi_2 = \begin{bmatrix} YAT + YF_k^T(x^{(i)}) + C_s^TB_s^T \end{bmatrix}, \quad \pi_3 = \begin{bmatrix} 0 & 0 \\ 0 & C_s^TB_s^T \end{bmatrix}, \quad \pi_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ A_s^T X + F_k^T(x^{(i)}) X + \theta C_s^TB_s^T \end{bmatrix}, \quad \theta = \begin{bmatrix} \pi_5 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$
\[
\begin{align*}
\pi_5 &= \begin{bmatrix}
0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \\
\hat{\mu} &= \begin{bmatrix}
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}^T, \\
\pi_6 &= \begin{bmatrix}
0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \\
\pi_7 &= \begin{bmatrix}
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \\
H &= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^T, \\
\pi_8 &= \begin{bmatrix}
D^T & D^TX
\end{bmatrix}, \\
Z &= \begin{bmatrix}
\pi_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \\
\pi_9 &= \begin{bmatrix}
EY & E
\end{bmatrix},
\end{align*}
\]

then the closed-loop system (2.4) is FTSB by the dynamic output feedback controller (2.3) with the parameters

\[
A_c = R^{-1}(A_s - XAY - XBC_s - \theta B_c CY)N^{-T}, \quad B_c = R^{-1}B_s, \quad C_c = C_sN^{-T},
\]

and \( R \) and \( N \) are nonsingular matrices such that

\[
RN^T = I - XY.
\]

**Proof.** According to Lemma 2.1, the condition (3.1) is equivalent to the inequality

\[
\nu_k^T P^{-1} \nu_k \leq 1, \quad k = 1, 2, \cdots, q.
\]

Considering the ellipsoid

\[
\Xi := \{ \eta \in R^n : \eta^T P \eta \leq 1 \},
\]

the relations (3.1), (3.2), (3.6) and using the approach [1, 3], we obtain

\[
\Omega_0 \subset \Xi \subset \Omega.
\]

If \( V \) is the Lyapunov function

\[
V[\eta(k), w(k)] = \eta^T(k)P\eta(k) + w^T(k)Qw(k),
\]

then condition (3.3) shows that for all \( \eta(k) \in \Omega, k = 1, 2, \cdots, M \) the inequality

\[
\mathcal{E}[V[\eta(k + 1), w(k + 1)] - \beta V[\eta(k), w(k)]] < 0
\]

holds. Indeed, substituting (2.4) in (3.9) and using the relations

\[
\begin{align*}
\mathcal{E}\{a(k) - a\} &= 0, \\
\mathcal{E}\{\theta(k) - \theta\} &= 0, \\
\mathcal{E}\{(a(k) - a)^2\} &= \alpha(1 - \alpha), \\
\mathcal{E}\{(\theta(k) - \theta)^2\} &= \theta(1 - \theta),
\end{align*}
\]

we can estimate the left-hand side of (3.9) as

\[
\begin{align*}
\mathcal{E}\{V[\eta(k + 1), w(k + 1)] - \beta V[\eta(k), w(k)]\}
&= \mathcal{E}\{\eta^T(k)\bar{A}^T \bar{P} \bar{A} \eta(k) + \eta^T(k)\bar{A}^T \bar{P} \bar{B}w(k) + \alpha(1 - \alpha)\eta^T(k)\Delta \bar{A}^T \bar{P} \Delta \bar{A} \eta(k) \\
&\quad + \theta(1 - \theta)\eta^T(k)\bar{A}^T \bar{P} \bar{A} \eta(k) + w^T(k)\bar{B}^T \bar{P} \bar{A} \eta(k) + w^T(k)\bar{B}^T \bar{P} \bar{B}w(k) \\
&\quad + w^T(k)G^T Qw(k) - \beta \eta^T(k)P \eta(k) - \beta w^T(k)Qw(k)\}
&= \zeta^T(k)\Pi \zeta(k) < 0,
\end{align*}
\]

where

\[
\Pi = \begin{bmatrix}
\mathcal{E}\{\eta^T(k)\bar{A}^T \bar{P} \bar{A} \eta(k)\} & \mathcal{E}\{\eta^T(k)\bar{A}^T \bar{P} \bar{B}w(k)\} \\
\mathcal{E}\{\eta^T(k)\bar{A}^T \bar{P} \bar{A} \eta(k)\} & \mathcal{E}\{\eta^T(k)\bar{A}^T \bar{P} \bar{B}w(k)\}
\end{bmatrix}.
\]
where \( \zeta(k) = [\eta^T(k) \ w^T(k)]^T \) and

\[
\Pi = \begin{bmatrix}
\psi & \tilde{A}^T \tilde{P} \tilde{B} \\
\star & \tilde{B}^T \tilde{P} \tilde{B} + G^T Q G - \beta Q
\end{bmatrix},
\]

\[
\psi = \tilde{A}^T \tilde{P} \tilde{A} + \alpha(1 - \alpha) \Delta \tilde{A}^T \tilde{P} \Delta \tilde{A} + \theta(1 - \theta) \tilde{A}^T \tilde{P} \tilde{A} - \beta P.
\]

It follows from Lemma 2.1 that the inequality \( \Pi < 0 \) holds if and only if

\[
\begin{bmatrix}
-\beta P & 0 & \tilde{A}^T P & \sqrt{\alpha(1 - \alpha)} \Delta \tilde{A}^T P & \sqrt{\theta(1 - \theta)} \tilde{A}^T P & 0 \\
\star & -\beta Q & \tilde{B}^T P & 0 & 0 & G^T Q \\
\star & \star & -P & 0 & 0 & 0 \\
\star & \star & \star & -P & 0 & 0 \\
\star & \star & \star & \star & -P & 0 \\
\star & \star & \star & \star & \star & -Q
\end{bmatrix} < 0. \tag{3.11}
\]

The inequality (3.3) implies that

\[
\pi_1 = \begin{bmatrix}
-Y & -I \\
\star & -X
\end{bmatrix} < 0, \tag{3.12}
\]

and according to Lemma 2.1, the condition (3.12) is equivalent to the inequality \( X - Y^{-1} > 0 \). Therefore, \( I - XY \) is a nonsingular matrix and there are nonsingular matrices \( R \) and \( N \) such that (3.5) holds. Considering the matrices

\[
\Gamma_1 = \begin{bmatrix}
Y & I \\
N^T & 0
\end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix}
I & X \\
0 & R^T
\end{bmatrix}, \tag{3.13}
\]

we note that they are also non-singular and we can set

\[
P := \Gamma_2 \Gamma_1^{-1}. \tag{3.14}
\]

It follows from (3.5), (3.13), (3.14) that

\[
P = \begin{bmatrix}
X & R \\
\star & N^{-1} Y (X - Y^{-1}) Y N^{-1}
\end{bmatrix},
\]

and since

\[
N^{-1} Y (X - Y^{-1}) Y N^{-1} > 0,
\]

\[
N^{-1} Y (X - Y^{-1}) Y N^{-1} - R^T X^{-1} R
\]

\[
=R^T (Y X - I)^{-1} (X - Y^{-1}) (Y X - I)^{-1} R > 0,
\]

we have \( P > 0 \).
Multiplying (3.11) by the matrices \( \text{diag} \{ \Gamma_1^T, I, \Gamma_1^T, \Gamma_1^T, I, I \} \) and its transpose from the left and from the right, respectively, and using (3.14), we obtain

\[
\begin{bmatrix}
-\beta \Gamma_1^T \Gamma_2 & 0 & \Gamma_1^T \bar{A}^T \Gamma_2 & \sqrt{\alpha(1-\alpha)} \Gamma_1^T \Delta \bar{A}^T \Gamma_2 & \sqrt{\theta(1-\theta)} \Gamma_1^T \bar{A}^T \Gamma_2 & 0 \\
* & -\beta Q & B^T \Gamma_2 & 0 & 0 & G^T Q \\
* & * & -\Gamma_1^T \Gamma_2 & 0 & 0 & 0 \\
* & * & * & -\Gamma_1^T \Gamma_2 & 0 & 0 \\
* & * & * & * & -\Gamma_1^T \Gamma_2 & 0 \\
* & * & * & * & * & -Q \\
\end{bmatrix} < 0. \tag{3.15}
\]

Introducing the matrices

\[
A_s := XAY + XBC_s + \theta B_s CY + RA_s N^T, \quad B_s := RB_c, \quad C_s := C_s N^T,
\]

we write (3.15) as

\[
\Psi + \theta \mu^T + \mu \theta^T < 0, \tag{3.16}
\]

where

\[
\Psi = \begin{bmatrix}
\beta \pi_1 & 0 & \pi_2 + \alpha \Delta \pi & \sqrt{\alpha(1-\alpha)} \Delta \pi & \sqrt{\theta(1-\theta)} \pi_3 & 0 \\
* & -\beta Q & \pi_4 & 0 & 0 & G^T Q \\
* & * & \pi_1 & 0 & 0 & 0 \\
* & * & * & \pi_1 & 0 & 0 \\
* & * & * & * & \pi_1 & 0 \\
* & * & * & * & * & \pi_6 \\
\end{bmatrix},
\]

\[
\pi_2 = \begin{bmatrix}
Y A^T + Y F_k^T(x) + C_s^T B^T & A_s^T \\
A^T + F_k^T(x) & A^T X + F_k^T(x) X + \theta C^T B_s^T \\
\end{bmatrix},
\]

\[
\Delta \pi = \begin{bmatrix}
Y \Delta A^T \\
\Delta A^T X \\
\Delta A^T X \\
\end{bmatrix},
\]

\[
\mu = \begin{bmatrix} 0 & 0 & \pi_6 & 0 & \pi_7 & 0 \end{bmatrix}^T, \quad \pi_6 = \begin{bmatrix} 0 & F_k^T(x) X \end{bmatrix}.
\]

It follows from Lemma 2.2 that

\[\theta \mu^T + \mu \theta^T \leq \theta \theta^T + \mu \mu^T.\]

Therefore, if

\[
\Psi + \theta \theta^T + \mu \mu^T < 0, \tag{3.17}
\]

then (3.16) holds.

By Lemma 2.1, the condition of (3.17) is equivalent to the inequality

\[
\begin{bmatrix}
\Psi & \mu & \theta \\
* & -I & 0 \\
* & * & -I \\
\end{bmatrix} < 0. \tag{3.18}
\]
To deal with the parameter uncertainties $\Delta A$, we write (3.18) as

$$\Theta + HF(k)Z + Z^TF(k)H^T < 0,$$

where

$$\Theta = \begin{bmatrix} \psi_1 & \mu & \bar{\theta} \\ * & -I & 0 \\ * & * & -I \end{bmatrix}, \quad \psi_1 = \begin{bmatrix} \beta \pi_1 & 0 & \pi_2 & 0 & \sqrt{\theta(1-\theta)}\pi_3 & 0 \\ * & -\beta Q & \pi_4 & 0 & 0 & G^TQ \\ * & * & \pi_1 & 0 & 0 \\ * & * & * & \pi_1 & 0 \\ * & * & * & * & -Q \end{bmatrix}.$$  

It follows from Lemma 2.3 that

$$\begin{bmatrix} \Theta & \varepsilon H & Z^T \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0. \quad (3.20)$$

Indeed, we consider the left-hand side of (3.20) as the affine function of the state variable. Therefore, it is negative definite on $\Omega$ if so is it on the polytope vertices. Hence the condition (3.3) yields the inequality (3.20).

Successively using the inequality (3.9) with $\beta \geq 1$, we obtain

$$\mathcal{E}\{V[\eta(k),w(k)]\} < \beta^k \mathcal{E}\{V[\eta(0),w(0)]\}, \quad k = 1, 2, \ldots, M. \quad (3.21)$$

Therefore,

$$\beta^k \mathcal{E}\{V[\eta(0),w(0)]\} = \beta^k \mathcal{E}\{[\eta^T(0)P\eta(0) + w^T(0)Qw(0)]\} \leq \beta^k \lambda_{\max}(P)\|\eta(0)\|^2 + \beta^k \lambda_{\max}(Q)d$$

$$\leq \beta^M \lambda_{\max}(P)^{\max_i \|\eta^{(i)}_{\Omega_0}\|^2} + \beta^M \lambda_{\max}(Q)d. \quad (3.22)$$

On the other hand, since

$$\mathcal{E}\{V[\eta(k),w(k)]\} = \mathcal{E}\{[\eta^T(k)P\eta(k) + w^T(k)Qw(k)]\} \geq \mathcal{E}\{[\eta^T(k)P\eta(k)\}$$

we can combine the previous estimates to arrive at the inequality

$$\mathcal{E}\{[\eta^T(k)P\eta(k)\} < \beta^M \lambda_{\max}(P)^{\max_i \|\eta^{(i)}_{\Omega_0}\|^2} + \beta^M \lambda_{\max}(Q)d.$$

The condition (3.4) now yields

$$\mathcal{E}\{[\eta^T(k)P\eta(k)\} < 1, \quad k = 1, 2, \ldots, M.$$

Hence, for $0 \leq k \leq M$ any trajectory starting within the set $\Omega_0$ does not go out of the ellipsoid $\Xi$ and since $\Xi$ is a subset of $\Omega$, the proof is finished. \qed
Remark 3.1. For a given \( \beta \geq 1 \), we replace (3.4) by the inequalities
\[
\begin{bmatrix}
P & I \\
I & \lambda I
\end{bmatrix} < 0, \\
\begin{bmatrix}
Q & I \\
I & \gamma I
\end{bmatrix} < 0, \\
\begin{bmatrix}
\frac{1}{\sqrt{\beta^M c}} & \frac{1}{\sqrt{\beta^M d}} \\
\lambda & 0 \\
0 & \gamma
\end{bmatrix} \geq 0,
\]
where \( c = \max \| \eta^{(i)}_{\Omega_0} \|^2 \). Then the conditions of Theorem 3.1 can be converted into a convex feasibility problem involving linear matrix inequalities (LMIs) (3.1)-(3.3) and (3.24).

For \( w(k) = 0 \) Theorem 3.1 can be specified as follows.

**Corollary 3.1.** Let \( \Omega_0, \Omega \) be polytopes (2.5) and \( M \) a positive integer. If there are a real number \( \beta \geq 1 \) and matrices \( P > 0, X > 0, Y > 0, A_s, B_s, C_s \) such that
\[
\begin{bmatrix}
1 & Y^T_k \\
* & P
\end{bmatrix} \geq 0, \quad k = 1, 2, \ldots, q, 
\]
\[
\eta^{(i)}_{\Omega_0} P \eta^{(i)}_{\Omega_0} \leq 1, \quad i = 1, 2, \ldots, p, 
\]
\[
\begin{bmatrix}
\hat{\Theta} & \epsilon \hat{H} & 2^T \\
* & -\epsilon I & 0 \\
* & * & -\epsilon I
\end{bmatrix} < 0,
\]
\[
\beta^M \lambda_{\max} (P) \max_i \| \eta^{(i)}_{\Omega_0} \|^2 \leq 1,
\]
where
\[
\hat{\Theta} = \begin{bmatrix}
\hat{\Psi}_1 & \hat{\mu} & \hat{\theta} \\
* & -I & 0 \\
* & * & -I
\end{bmatrix}, \\
\hat{\Psi}_1 = \begin{bmatrix}
\beta \pi_1 & \bar{\pi}_2 & 0 & \sqrt{\theta(1-\theta)} \pi_3 \\
* & \pi_1 & 0 & 0 \\
* & * & \pi_1 & 0 \\
* & * & * & \pi_1
\end{bmatrix},
\]
\[
\pi_1 = \begin{bmatrix}
-Y & -I \\
* & -X
\end{bmatrix}, \\
\bar{\pi}_2 = \begin{bmatrix}
Y A^T + Y F^T_k (x^{(i)}) + C_s B_s^T & A_s^T \\
A^T + F^T_k (x^{(i)}) & A_s^T X + F^T_k (x^{(i)}) X + \theta C_s B_s^T
\end{bmatrix},
\]
\[
\pi_3 = \begin{bmatrix}
0 & 0 \\
0 & C_s B_s^T
\end{bmatrix}, \\
\hat{\theta} = \begin{bmatrix}
\pi_5 & 0 & 0 & 0 \\
\pi_5 & 0 & 0 & 0 \\
\pi_5 & 0 & 0 & 0 \\
Y & 0
\end{bmatrix},
\]
\[
\hat{\mu} = \begin{bmatrix}
0 & \bar{\pi}_6 & 0 & \pi_7
\end{bmatrix}^T, \\
\bar{\pi}_6 = \begin{bmatrix}
0 & F^T_k (x^{(i)}) X \\
0 & \sqrt{\theta(1-\theta)} C_s B_s^T
\end{bmatrix},
\]
\[
\hat{H} = \begin{bmatrix}
0 & \sqrt{\alpha(1-\alpha)} \pi_8 & 0 & 0 & 0 \\
0 & \sqrt{\alpha(1-\alpha)} \pi_8 & 0 & 0 & 0 \\
0 & \sqrt{\alpha(1-\alpha)} \pi_8 & 0 & 0 & 0 \\
D^T & D^T X
\end{bmatrix},
\]
\[
\hat{Z} = \begin{bmatrix}
\pi_9 & 0 & 0 & 0 \\
\pi_9 & 0 & 0 & 0 \\
\pi_9 & 0 & 0 & 0 \\
E Y & E
\end{bmatrix},
\]
then the closed-loop system (2.4) is FTSS by the dynamic output feedback controller (2.3) with the parameters
\[
A_c = R^{-1}(A_s - X A Y - X B C_s - \theta B_s C Y) N^{-T}, \\
B_c = R^{-1} B_s, \\
C_c = C_s N^{-T},
\]
and with nonsingular matrices \( R \) and \( N \) such \( R N^T = I - X Y \).
4. Example

Consider a discrete-time nonlinear quadratic systems described by (2.1) with the following parameters:

\[
A = \begin{bmatrix} 0.44 & -0.5 \\ -0.6 & -0.81 \end{bmatrix}, \quad B = \begin{bmatrix} 0.01 & 0 \\ 9.2 & 2.8 \end{bmatrix}, \quad C = \begin{bmatrix} 1.13 & 0.21 \\ 0.28 & 0.33 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]
\[
D_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0.1 & 0 \\ -1 & -0.2 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & -0.2 \\ -0.3 & 0 \end{bmatrix}.
\]

Choose the polytopes \( \Omega = [-2, 2] \times [-2, 2], \Omega_0 = [-0.5, 0.5] \times [-0.5, 0.5] \) and the initial condition \( x_0 = [0.4, -0.3]^T \). Moreover, we set \( d = 1.6, \beta = 2.5, \theta = 0.5, \alpha = 0.3, \epsilon = 0.8, M = 30 \). Using the Matlab LMI Toolbox, we determine the feasible solutions:

\[
P = \begin{bmatrix} 1.2563 & 0 \\ 0 & 1.2669 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.2184 & -0.0005 \\ -0.0005 & 0.2185 \end{bmatrix},
\]
\[
X = \begin{bmatrix} 0.5567 & 0.0147 \\ 0.0147 & 0.5824 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.2628 & -0.0026 \\ -0.0026 & 0.2449 \end{bmatrix},
\]
\[
A_s = \begin{bmatrix} 0.1830 & -0.2330 \\ -0.1968 & -0.3372 \end{bmatrix}, \quad B_s = \begin{bmatrix} 0.0556 & 0.0410 \\ -0.0027 & -0.1167 \end{bmatrix},
\]
\[
C_s = \begin{bmatrix} 3.1752 & 6.2579 \\ 6.5432 & 7.0615 \end{bmatrix}.
\]

Take the following matrix

\[
R = \begin{bmatrix} 0.11 & 0.05 \\ 0 & 0.13 \end{bmatrix}.
\]

Then (3.5) yields

\[
N = \begin{bmatrix} 7.7696 & -0.0181 \\ -3.0174 & 6.5953 \end{bmatrix}.
\]

Therefore, the gain matrices of the sought dynamic output feedback controller (2.3) are:

\[
A_c = \begin{bmatrix} 0.3228 & 0.2280 \\ -0.5255 & -0.1611 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0.5150 & 0.7728 \\ -0.0208 & -0.8980 \end{bmatrix},
\]
\[
C_c = \begin{bmatrix} 2.9591 & -10.2087 \\ -3.3892 & 11.6500 \end{bmatrix}.
\]

Fig. 1 shows the state trajectories of the open-loop systems, starting from the initial condition \( x_0 = [0.4, -0.3]^T \) in \( \Omega_0 \). It is clear that the open-loop system is not FTSB. Fig. 2 shows that for the time-interval \([0,30]\), the state trajectories of the closed-loop system starting the polytope \( \Omega_0 \) stay in the ellipsoid \( \Xi \). The state estimation of the system is presented in Fig. 3. By simulation results, the closed-loop system (2.4) is FTSB by the dynamic output feedback controller (2.3). The figures show the effectiveness of our method.
Figure 1: State response of the open-loop system.

Figure 2: State response of closed-loop system.

Figure 3: State estimate of closed-loop system.
5. Conclusions

We deal with finite-time dynamic output feedback control for a class of discrete-time nonlinear quadratic systems with stochastic parametric uncertainty, exogenous disturbance and missing measurements. The sufficient condition for FTSB under a dynamic output feedback controller are provided. As a consequence, a sufficient condition for FTSS is derived. A numerical example demonstrates the effectiveness of the method.

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