Probability-Guaranteed $H_\infty$-Filtering for 2D-Systems with Random Nonlinearities and Missing Measurements

Shanqiang Li$^{1,2}$, Xiuyan Peng$^{1,*}$ and Yujing Shi$^2$

$^1$College of Automation, Harbin Engineering University, Harbin 150001, PR. China.
$^2$Department of Applied Mathematics, Harbin University of Science and Technology, Harbin 150080, PR. China.

Received 1 October 2018; Accepted (in revised version) 23 November 2018.

Abstract. The probability-guaranteed $H_\infty$-filtering problem for 2D-discrete time systems with random nonlinearities and missing measurements is studied. To characterise the statistical uncertain parameters, the uniform distribution is used. The filter parameters are derived by LMI technique and a special parameter-box is used to guarantee $H_\infty$-performance with pre-specified probability constraints. An example shows the effectiveness of the filtering scheme developed.

AMS subject classifications: 93E11

Key words: Two-dimensional system, probability performance, $H_\infty$-filtering.

1. Introduction

Two-dimensional systems with information propagation in two independent directions have attracted considerable attention due to applications in image data processing and transmission, biomedical imaging processing, multidimensional digital filtering, thermal processes, and water stream heating [2, 6, 10]. In particular, for 2D-systems asymptotic and exponential stability are studied in [24], controller design problems in [4, 16], stability analysis and stabilisation of switched systems in [1, 21], and $H_\infty$-control problems for Markovian jump systems with state-delays and defective mode information in [20]. There are also results concerning the $H_\infty$-filtering design approach — e.g. the problem of $H_\infty$-filtering is solved for uncertain discrete systems [11], for uncertain continuous systems [22], for time delay systems [3, 12] and for stochastic systems [7].

It is worth noting that all the above-mentioned results are based on the assumption that sensor measurements are perfect. However, in practical situations, it is possible that signals are measured during their transmission and some measurements are missing. Moreover,
the random nonlinearities often occurring in networked environments, cause a particular concern. Therefore, the stability analysis, controller synthesis and filtering design for 1D networked control systems are widely studied \[5, 9, 18\]. On the other hand, there are only a few results available for two-dimensional network-based systems — e.g. robust $H_{\infty}$-filtering with random mixed delays and with intermittent measurements is, respectively, considered in \[15\] and \[14\], recursive filtering for nonlinear systems with measurement degradation in \[17\], and the state estimation in complex networks with randomly occurring nonlinearities and randomly varying sensor delays in \[13\].

In traditional control theory, it is rarely possible to achieve the prescribed performance of control systems with the probability 1, but it is acceptable to realise the prescribed performance objectives with an expected probability. The probability guaranteed robust $H_{\infty}$-controller design method was proposed by Yaesh et al. \[23\]. Subsequently, this method found applications in filtering design problems. Hu et al. \[8\] studied the probability-guaranteed $H_{\infty}$-finite-horizon filtering for nonlinear time-varying systems, Wei et al. \[19\] considered the probability-guaranteed set-membership filtering problems for time-varying systems with incomplete measurements. Nevertheless, to the best of the authors’ knowledge, for 2D-systems the probability-guaranteed $H_{\infty}$-filtering problems have not yet been studied.

Here, we deal with probability-guaranteed $H_{\infty}$-filtering problems for discrete-time 2D-systems with randomly occurring nonlinearities and missing measurements. The main features of this work are:

1. The comprehensive consideration of the systems, including uncertainty parameters, randomly occurring nonlinearities and incomplete measurement information.
2. The development of an $H_{\infty}$-filter. In particular, we construct a parameter-box, such that $H_{\infty}$-performance requirement is guaranteed with pre-specified probability constraints.
3. The proof of the stochastic mean-square asymptotical stability of the 2D-filtering error systems.

2. Problem Formulation

We consider the following uncertain nonlinear discrete-time 2D-system:

\[
\begin{bmatrix}
    x_h(i+1,j) \\
    x_v(i,j+1)
\end{bmatrix} = A(\theta)x(i,j) + \alpha(i,j)g(x(i,j)) + B(\theta)\omega(i,j),
\]

\[z(i,j) = Mx(i,j),\]

where $\omega(i,j) \in \mathbb{R}^{n_{\omega}}$ is the disturbance input in $l_2[0, \infty)$, $z(i,j) \in \mathbb{R}^{n_z}$ the controlled output, $M$ a known real-valued matrix of appropriate dimension, and

\[x(i,j) := \begin{bmatrix} x_h(i,j) \\ x_v(i,j) \end{bmatrix} \in \mathbb{R}^{n_1+n_2}\]
with the horizontal and vertical states $x^h(i, j) \in \mathbb{R}^{n_1}$ and $x^v(i, j) \in \mathbb{R}^{n_2}$, respectively.

According to [8, 19, 23], the uncertainty matrices $A(\theta), B(\theta)$ have the form

$$
A(\theta) = A_0 + \sum_{k=1}^{q} \theta_k A_k, \quad B(\theta) = B_0 + \sum_{k=1}^{q} \theta_k B_k,
$$

where $A_0, B_0$ can be considered as nominal system matrices, $A_k, B_k, k = 1, \ldots, q$ are known constant matrices and $\theta = [\theta_1, \theta_2, \ldots, \theta_q]^T \in \mathbb{R}^q$ is an uncertainty parameter vector. We assume that all $\theta_k, k = 1, \ldots, q$ are mutually independent random variables uniformly distributed over $[\delta_k, \sigma_k]$ with given $\delta_k$ and $\sigma_k$. Then the random variables $\theta_k, k = 1, \ldots, q$ represent the deviations of the system parameters from their nominal values. Since the uncertainty parameter vector $\theta$ belongs to a $q$-dimensional hyper-rectangle $\mathbb{T}$ with the vertices

$$
V_{\mathbb{T}} = \{[\theta_1, \theta_2, \ldots, \theta_q]^T : \theta_k \in [\delta_k, \sigma_k], k = 1, 2, \ldots, q\},
$$

the uncertainty matrices in (2.1) are contained in the convex polytope

$$
\Omega = \left\{[A(\theta), B(\theta)] = \sum_{l=1}^{2^q} f_l \Omega^{(l)}, 0 \leq f_l \leq 1, \sum_{l=1}^{2^q} f_l = 1 \right\}, \quad (2.2)
$$

where $\Omega^{(l)} = [A^{(l)}, B^{(l)}], l = 1, 2, \ldots, 2^q$ are vertex matrices.

The nonlinear vector-valued function $g(x(i, j)) \in \mathbb{R}^{n_1+n_2}$ in (2.1) satisfies the following condition

$$
\|g(x(i, j))\|^2 = \|Ex(i, j)\|^2 \quad (2.3)
$$

with a known matrix $E$. The stochastic variable $a(i, j)$ is a Bernoulli distributed white sequence taking values of 0 or 1 with the probability

$$
\text{Prob}\{a(i, j) = 1\} = \bar{a}, \quad \text{Prob}\{a(i, j) = 0\} = 1 - \bar{a},
$$

where $0 \leq \bar{a} \leq 1$ is a known constant.

Here we consider the measurement output $y(i, j)$ of the system (2.1) with the missing measurements of the form

$$
y(i, j) = \Xi(i, j)Cx(i, j) + D\omega(i, j) = \sum_{k=1}^{n_y} \beta_k(i, j)C_kx(i, j) + D\omega(i, j), \quad (2.4)
$$

where $y(i, j) \in \mathbb{R}^{n_y}$ is the actual measurement output. The stochastic variables $\beta_k(i, j), k = 1, 2, \ldots, n_y$, responsible for missing measurements, are the Bernoulli distributed white sequences taking values of 0 or 1 with the probability

$$
\text{Prob}\{\beta_k(i, j) = 1\} = \bar{\beta}_k, \quad \text{Prob}\{\beta_k(i, j) = 0\} = 1 - \bar{\beta}_k,
$$

where $0 \leq \bar{\beta}_k \leq 1, k = 1, 2, \ldots, n_y$ are known constants. Throughout the paper, we assume that $\theta_k, k = 1, \ldots, p$ and $a(i, j), \beta_k(i, j), k = 1, 2, \ldots, n_y$ are uncorrelated random
variables. We also set \( \Xi(i,j) := \text{diag}\{\beta_1(i,j), \beta_2(i,j), \ldots, \beta_{n_y}(i,j)\} \), \( \hat{\Xi} := E\{\Xi(i,j)\} = \text{diag}\{\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_{n_y}\} \) and \( C_k := \text{diag}\{0 \ldots 0 1 \ldots 0\} C \).

For 2D-systems (2.1), we also adopt a based on the measurement information (2.4) filter — viz.

\[
\begin{bmatrix}
    x^h_c(i+1,j) \\
    x^v_c(i,j+1) \\
    x^h_c(i+1,j) \\
    x^v_c(i,j+1)
\end{bmatrix} = \begin{bmatrix}
    A_c & B_c & y(i,j), \\
    C_c & \end{bmatrix} = A_c x_c(i,j) + B_c y(i,j),
\]

(2.5)

where \( z_c(i,j) \in \mathbb{R}^{n_c} \) is the estimated output,

\[
x_c(i,j) = \begin{bmatrix}
    x^h_c(i,j) \\
    x^v_c(i,j)
\end{bmatrix} \in \mathbb{R}^{n_1+n_2},
\]

\( x^h_c(i,j) \in \mathbb{R}^{n_1}, x^v_c(i,j) \in \mathbb{R}^{n_2} \) are the state estimates, and \( A_c, B_c, C_c \) the filter parameters to be designed. Thus the filtering error has the form

\[
\begin{bmatrix}
    x^h(i+1,j) \\
    x^v(i,j+1) \\
    x^h(i+1,j) \\
    x^v(i,j+1)
\end{bmatrix} = \tilde{A}(\theta)\tilde{x}(i,j) + \tilde{B}\tilde{x}(i,j) + (\tilde{a} + \tilde{a}(i,j))G(x(i,j)) + \tilde{B}(\theta)\omega(i,j),
\]

(2.6)

where

\[
\tilde{A}(\theta) = \begin{bmatrix}
    A(\theta) & 0 \\
    B_c \tilde{\Xi} & A_c
\end{bmatrix}, \quad \tilde{B}(\theta) = \begin{bmatrix}
    0 & 0 \\
    B_c \tilde{\Xi}(i,j)C & 0
\end{bmatrix},
\]

\[
\tilde{B}(\theta) = \begin{bmatrix}
    B(\theta) \\
    B_c D
\end{bmatrix}, \quad G(x(i,j)) = \begin{bmatrix}
    g(x(i,j)) \\
    0
\end{bmatrix}, \quad \tilde{x}(i,j) = \begin{bmatrix}
    x(i,j) \\
    x_c(i,j)
\end{bmatrix},
\]

\[
\hat{\Xi}(i,j) = \Xi(i,j) - \tilde{\Xi}, \quad \hat{\Xi}(i,j) = \Xi(i,j) - \tilde{\Xi}, \quad \hat{\Xi}(i,j) = \Xi(i,j) - \tilde{\Xi}, \quad \hat{\Xi}(i,j) = \Xi(i,j) - \tilde{\Xi}, \quad \hat{\Xi}(i,j) = \Xi(i,j) - \tilde{\Xi}\]

\[
\tilde{\xi}(i,j) = z(i,j) - z_c(i,j), \quad \tilde{M} = \begin{bmatrix}
    M & -C_c
\end{bmatrix}.
\]

We now also assume that the boundary condition in system (2.6) satisfies the relation

\[
\lim_{N \to \infty} E \left\{ \sum_{k=0}^{N} \left[ \left\| x^h(0,k) \right\| + \left\| x^v(k,0) \right\|^2 \right] \right\} < \infty.
\]

(2.7)

**Definition 2.1.** The filtering error system (2.6), satisfying the condition (2.7) is called mean-square asymptotically stable if for \( \omega(i,j) = 0 \) one has

\[
\lim_{i+j \to \infty} E \left\{ \left\| \tilde{x}(i,j) \right\|^2 \right\} = 0.
\]
Our aim now is to find a filter (2.5), such that for a probability \( 0 < p < 1 \) and a specified disturbance attenuation level \( \gamma > 0 \), one has

\[
\text{Prob} \{ J < 0 \} \geq p,
\]

where

\[
J = E \left\{ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (\| \tilde{z}(i,j) \|^2 - \gamma^2 \| \omega(i,j) \|^2) \right\}.
\]

More specifically, we are looking for dynamic output feedback controller matrices \( A_c, B_c, C_c \) in (2.5) and a parameter-box \( \Upsilon_T \subset \Upsilon \), which simultaneously satisfy the following conditions:

A. The probability of \( \theta \in \Upsilon_T \) is greater to or equal to \( p \).

B. For the parameter-box \( \Upsilon_T \) generated by \( \theta_k \in [a_k, b_k] \subset [\delta_k, \sigma_k], k = 1, 2, \cdots, q \) with the vertices

\[
V_{\Upsilon_T} = \{ [\theta_1, \theta_2, \cdots, \theta_q]^T \mid \theta_k \in [a_k, b_k], k = 1, 2, \cdots, q \}
\]

the \( H_\infty \)-performance requirement \( J < 0 \) can be guaranteed.

### 3. The Main Result

We want to develop a probability-guaranteed \( H_\infty \)-filters for 2D-systems (2.1) with random nonlinearities and missing measurements.

Let us start with the condition A. Following considerations [8, 19, 23], the probability constraint for \( \theta \in \Upsilon_T \) can be written as

\[
\prod_{k=1}^{q} (b_k - a_k) \geq \bar{p},
\]

where \( \bar{p} = p \prod_{k=1}^{q} (\sigma_k - \delta_k) \) and \( a_k, b_k, k = 1, \cdots, q \) are parameters to determine.

**Lemma 3.1.** For a fixed \( p > 0 \), the inequality (3.1) is equivalent to the estimate

\[
\prod_{l=1}^{m_1} s_{1,l} \geq \sqrt{\bar{p}},
\]

where \( s_{1,l}, l = 1, \cdots, m_1 \) are positive scalars we have to determine. If \( q \) is even, we set \( m_1 = q/2 \), so that

\[
\begin{bmatrix}
    b_{2l-1} - a_{2l-1} & s_{1,l} \\
    * & b_{2l} - a_{2l}
\end{bmatrix} \geq 0, \quad l = 1, \cdots, m_1.
\]

When \( q \) is odd, we set \( m_1 = (q - 1)/2 + 1 \) and the inequalities (3.3) take the form

\[
\begin{bmatrix}
    b_q - a_q & s_{1,m_1} \\
    * & 1
\end{bmatrix} \geq 0, \quad l = 1, 2, \cdots, m_1 - 1.
\]
Thus, according to Lemma 3.1, the probability constraint (3.1) can be converted into (3.2)-(3.4), which is easier to handle.

### 3.1. Probability-guaranteed $H_\infty$-performance

We now consider condition B. The theorem below provides sufficient conditions for the filtering error system (2.6) to satisfy $H_\infty$-performance.

**Theorem 3.1.** Assume that a disturbance attenuation level $\gamma > 0$, probability constraint $p > 0$, and filter parameters $A_c$, $B_c$, $C_c$ are given and there are $\varepsilon > 0$ and a positive definite matrix $P = \text{diag} \{p^h, p^\nu, p^h, p^\nu\}$ such that

$$
\bar{\Omega} = \begin{bmatrix}
\Xi & \ast & \ast \\
\bar{A}^T(\theta)P\bar{A}(\theta) & \bar{A}^2P + \bar{a}(1-\bar{a})P - \varepsilon H_1^T H_1 & \ast \\
\bar{B}^T(\theta)P\bar{A}(\theta) & \bar{A}\bar{B}^T(\theta)P & \bar{B}^T(\theta)P\bar{B}(\theta) - \gamma^2 I
\end{bmatrix} < 0,
$$

where

$$
\Xi = \bar{A}^T(\theta)P\bar{A}(\theta) + \bar{B}^T(I_{n_y} \otimes P)\bar{B} - P + \bar{M}^T \bar{M} + \varepsilon H_2^T E^T E H_2,
$$

$$
H_1 = [I \ I], \quad H_2 = [I \ 0],
$$

$$
\bar{B} = \left[ \sqrt{\beta_1 (1 - \beta_1)} \bar{B}_1^T, \sqrt{\beta_2 (1 - \beta_2)} \bar{B}_2^T, \ldots, \sqrt{\beta_{n_y} (1 - \beta_{n_y})} \bar{B}_{n_y}^T \right]^T,
$$

$$
\hat{B}_k = \begin{bmatrix}
0 & 0 \\
B_k & 0
\end{bmatrix}, \quad k = 1, \ldots, n_y.
$$

Then the uncertain 2D-system (2.1) satisfies the $H_\infty$-performance requirement $J < 0$.

**Proof.** We first prove the mean-square asymptotical stability of the 2D-filtering error system (2.6) with the disturbance $o(i, j) = 0$. Set

$$
\bar{J} = E \left\{ \tilde{x}'^T P \tilde{x}' | \tilde{x} \right\} - \tilde{x}^T P \tilde{x},
$$

where

$$
\tilde{x}' := \left[ x^h(i+1, j)^T, x^v(i, j+1)^T, x^h(i, j+1)^T, x^v(i+1, j)^T \right]^T,
$$

$$\tilde{x} := \tilde{x}(i, j) = \left[ x^h(i, j)^T, x^v(i, j)^T, x^h(i, j)^T, x^v(i, j)^T \right]^T.
$$

For simplicity, we write $x$ for $x(i, j)$ and use similar abbreviations for other expressions. Assuming that $\omega(i, j) = 0$ and substituting (2.6) into the Eq. (3.6), we obtain

$$
\bar{J} = E \left\{ [\hat{A}(\theta)\bar{x} + \bar{B}\bar{x} + (\bar{a} + \bar{a})G(x)]^T P [\hat{A}(\theta)\tilde{x} + \bar{B}\tilde{x} + (\bar{a} + \bar{a})G(x)] \right\} - \tilde{x}^T P \tilde{x},
$$

$$
\bar{J} = \tilde{x}'^T \hat{A}^T(\theta)P\hat{A}(\theta)\tilde{x} + 2\bar{a}\tilde{x}'^T \hat{A}^T(\theta)PG(x) + \tilde{x}'^T \hat{B}^T(I_{n_y} \otimes P)\bar{B}\tilde{x} + \bar{a}^2 G^T(x)PG(x)
$$

$$
+ \bar{a}(1-\bar{a})G^T(x)PG(x) - \tilde{x}'^T P \tilde{x} = \eta^T \hat{\Omega} \eta,
$$

(3.7)
where
\[ \eta = \begin{bmatrix} x^T & G(x) \end{bmatrix} \]
\[ \bar{\Omega} = \begin{bmatrix} \bar{A}^T(\theta)P\bar{A}(\theta) + \bar{B}^T(I_n \otimes \bar{B} - P \\
\bar{A}P\bar{A}(\theta) \end{bmatrix} \begin{bmatrix} \bar{a}^2P + \bar{a}(1 - \bar{a})P \end{bmatrix}. \]

The condition (2.3) yields that
\[ ||H_1G(x)||^2 \leq ||EH_2\bar{x}||^2, \quad (3.8) \]
and using (3.7) and (3.8), we obtain
\[ \hat{J} \leq E \left\{ \eta^T\bar{\Omega}\eta - \epsilon \left( ||H_1G(x)||^2 - ||EH_2\bar{x}||^2 \right) \right\} = E \left\{ \eta^T\bar{\Omega}\eta \right\}, \]
where
\[ \bar{\Omega} = \bar{\Omega} + \text{diag} \left\{ \epsilon H_2^2 E^TH_2, -\epsilon H_1^2 H_1 \right\}. \]

It follows from (3.5) that \( \bar{\Omega} < 0 \). Therefore, for all \( \eta \neq 0 \), we have
\[ \frac{\hat{J}}{\bar{x}^T \bar{P} \bar{x}} \leq -\frac{\eta^T(-\bar{\Omega})\eta}{\bar{x}^T \bar{P} \bar{x}} \leq -\frac{\lambda_{\min}(-\bar{\Omega})\eta^T\eta}{\lambda_{\max}(P)\bar{x}^T \bar{x}} \leq -\frac{\lambda_{\min}(-\bar{\Omega})}{\lambda_{\max}(P)} = \delta - 1, \quad (3.9) \]
where \( \delta = 1 - \frac{\lambda_{\min}(-\bar{\Omega})}{\lambda_{\max}(P)} \), and since \( \lambda_{\min}(-\bar{\Omega})/\lambda_{\max}(P) > 0 \), then \( \delta < 1 \). Taking into account the inequality (3.9), we observe that
\[ \delta \geq \frac{E \left\{ \bar{x}^T \bar{P} \bar{x} | \bar{x} \right\}}{\bar{x}^T \bar{P} \bar{x}} > 0, \]
and obtain \( 0 < \delta < 1 \). Thus
\[ E \left\{ \bar{x}^T \bar{P} \bar{x} | \bar{x} \right\} \leq \delta E \left\{ \bar{x}^T \bar{P} \bar{x} \right\}, \]
and we use this inequality to obtain
\[ E \left\{ x^T(k + 1, 0)^T P^x x^T(k, k + 1) + x_c^T(k + 1, 0)^T P_c^x x_c^T(k, k) \right\}, \]
\[ \vdots \]
\[ E \left\{ x^T(k, 0)^T P^x x^T(k, 0) + x_c^T(k, 0)^T P_c^x x_c^T(k, 0) \right\}, \]
\[ \vdots \]
Summing both sides of (3.10) leads to the inequality
\[
E \left\{ \sum_{j=0}^{k+1} \left[ x^h(k+1-j,j)^T P^h x^h(k+1-j,j) + x^\nu(k+1-j,j)^T P^\nu x^\nu(k+1-j,j) \right] \right\} 
+ x^h_c(k-j,j)^T P^h_c x^h_c(k-j,j) + x^\nu_c(k-j,j)^T P^\nu_c x^\nu_c(k-j,j) \right\} 
+ E \left\{ x^h(0,k+1)^T P^h x^h(0,k+1) + x^\nu(k+1,0)^T P^\nu x^\nu(k+1,0) 
+ x^h_c(0,k+1)^T P^h_c x^h_c(0,k+1) + x^\nu_c(k+1,0)^T P^\nu_c x^\nu_c(k+1,0) \right\}. \tag{3.11}
\]
Successive application of (3.11) shows that
\[
E \left\{ \sum_{j=0}^{k+1} \left[ x^h(k+1-j,j)^T P^h x^h(k+1-j,j) + x^\nu(k+1-j,j)^T P^\nu x^\nu(k+1-j,j) \right] \right\} 
+ x^h_c(k-j,j)^T P^h_c x^h_c(k-j,j) + x^\nu_c(k-j,j)^T P^\nu_c x^\nu_c(k-j,j) \right\} 
\leq E \left\{ \sum_{j=0}^{k+1} \delta^{j} \left[ x^h(0,k+1-j)^T P^h x^h(0,k+1-j) + x^\nu(k+1-j,0)^T P^\nu x^\nu(k+1-j,0) 
+ x^h_c(0,k+1-j)^T P^h_c x^h_c(0,k+1-j) + x^\nu_c(k+1-j,0)^T P^\nu_c x^\nu_c(k+1-j,0) \right] \right\}.
\]
This yields
\[
E \left\{ \sum_{j=0}^{k+1} \| \tilde{x}(k+1-j,j) \|^2 \right\} 
\leq \kappa E \left\{ \sum_{j=0}^{k+1} \delta^{j} \left[ \| \tilde{x}^h(0,k+1-j) \|^2 + \| \tilde{x}^\nu(k+1-j,0) \|^2 \right] \right\}, \tag{3.12}
\]
where
\[
\delta = \max \left\{ \lambda_{\max}(\bar{P}^h), \lambda_{\max}(\bar{P}^\nu) \right\},
\]
\[
\kappa = \frac{\lambda_{\max}(\bar{P})}{\lambda_{\min}(\bar{P})},
\]
\[
\tilde{x}^h = \begin{bmatrix} x^h \\ x^h_c \end{bmatrix}, \quad \tilde{x}^\nu = \begin{bmatrix} x^\nu \\ x^\nu_c \end{bmatrix},
\bar{P}^h = \text{diag} \left\{ p^h, p^h_c \right\}, \quad \bar{P}^\nu = \text{diag} \left\{ p^\nu, p^\nu_c \right\}.
\]
Using the notation $\chi_k = \sum_{j=0}^{k} \|\tilde{x}(k-j, j)\|^2$, we write the inequality (3.12) for various $k$, so that

$$
E\{\chi_0\} \leq \kappa E\left\{\|\tilde{x}(0,0)\|^2 + \|\tilde{y}(0,0)\|^2\right\},
$$

$$
E\{\chi_1\} \leq \kappa \left[\delta E\left\{\|\tilde{x}(0,0)\|^2 + \|\tilde{y}(0,0)\|^2\right\} + E\left\{\|\tilde{x}(0,1)\|^2 + \|\tilde{y}(1,0)\|^2\right\}\right],
$$

$$
\ldots
$$

$$
E\{\chi_N\} \leq \kappa \left[\delta^N E\left\{\|\tilde{x}(0,0)\|^2 + \|\tilde{y}(0,0)\|^2\right\} + \delta^{N-1} E\left\{\|\tilde{x}(0,1)\|^2 + \|\tilde{y}(1,0)\|^2\right\}
+ \ldots + E\left\{\|\tilde{x}(0,N)\|^2 + \|\tilde{y}(N,0)\|^2\right\}\right].
$$

(3.13)

Summing the inequalities (3.13), we find that

$$
\sum_{k=0}^{N-1} E\{\chi_k\} \leq \kappa \left(1 + \delta + \ldots + \delta^N\right) E\left\{\|\tilde{x}(0,0)\|^2 + \|\tilde{y}(0,0)\|^2\right\} + \kappa \left(1 + \delta + \ldots + \delta^{N-1}\right),
$$

$$
E\left\{\|\tilde{x}(0,1)\|^2 + \|\tilde{y}(1,0)\|^2\right\} + \ldots + \kappa E\left\{\|\tilde{x}(0,N)\|^2 + \|\tilde{y}(N,0)\|^2\right\}
\leq \kappa \left(1 + \delta + \ldots + \delta^N\right) E\left\{\|\tilde{x}(0,0)\|^2 + \|\tilde{y}(0,0)\|^2\right\} + \kappa \left(1 + \delta + \ldots + \delta^{N-1}\right),
$$

$$
E\left\{\|\tilde{x}(0,1)\|^2 + \|\tilde{y}(1,0)\|^2\right\} + \ldots + \kappa \left(1 + \delta + \ldots + \delta^N\right)
\times E\left\{\|\tilde{x}(0,N)\|^2 + \|\tilde{y}(N,0)\|^2\right\}
$$

$$
= \kappa \left[\frac{1 - \delta^N}{1 - \delta}\right] E\left\{\sum_{k=0}^{N-1} \|\tilde{x}(0,k)\|^2 + \|\tilde{y}(k,0)\|^2\right\}.
$$

(3.14)

The condition (2.7) ensures that the right-hand side of (3.14) is bounded. Therefore, \( \lim_{k \to \infty} E\{\chi_k\} = 0 \) or

$$
\lim_{i+j \to \infty} E\left\{\|\tilde{x}(i,j)\|^2\right\} = 0.
$$

Then by Definition 2.1, the closed-loop system (2.6) is mean-square asymptotically stable.

Now we can establish the \( H_{\infty} \)-performance for the filtering error system (2.6). Assuming zero initial boundary conditions, substituting (2.6) into (3.6) and considering (3.8), we obtain

$$
\tilde{J} = E\left\{\tilde{y}^T \hat{\Omega} \tilde{y} - \tilde{x}^T \tilde{z} + \gamma^2 \omega^T \omega\right\},
$$

where \( \tilde{y} = [\tilde{x}^T \quad G^T(x) \quad \sigma^T]^T \). Since \( \hat{\Omega} < 0 \) — cf. (3.5), we have

$$
E\left\{\tilde{x}^T P \tilde{x} | \tilde{x}\right\} < \tilde{x}^T P \tilde{x} - \tilde{z}^T \tilde{z} + \gamma^2 \omega^T \omega.
$$

(3.15)
Adding both sides of the inequality (3.16), we have

\[
E\left\{ x^\gamma(k + 1, 0)^T P^\gamma x^\gamma(k + 1, 0) + x^\gamma_k(k + 1, 0)^T P^\gamma x^\gamma_k(k + 1, 0) \right\}
\]

\[
= E\left\{ x^\gamma(k + 1, 0)^T P^\gamma x^\gamma(k + 1, 0) + x^\gamma_k(k + 1, 0)^T P^\gamma x^\gamma_k(k + 1, 0) \right\} + \sum_{j=0}^{k-1} \left[ x^\gamma(k + 1 - j, j)^T P^h x^\gamma(k + 1 - j, j) + x^\gamma(k + 1 - j, j)^T P^\gamma x^\gamma(k + 1 - j, j) \right]
\]

\[
\leq E\left\{ x^\gamma(k, 0)^T P^h x^\gamma(k, 0) + x^\gamma_k(k, 0)^T P^\gamma x^\gamma_k(k, 0) + x^\gamma_k(k, 0)^T P^h x^\gamma_k(k, 0)
\right.
\]

\[
+ x^\gamma(k, 0)^T P^\gamma x^\gamma(k, 0) - \bar{z}^T(k, 0)\bar{z}(k, 0) + \gamma^2 \omega^T(k, 0)\omega(k, 0) \right\} + E\left\{ x^\gamma(k, 1)^T P^h x^\gamma(k, 1) + x^\gamma(k, 1)^T P^\gamma x^\gamma(k, 1) \right\}
\]

\[
\leq E\left\{ x^\gamma(k, 0)^T P^h x^\gamma(k, 0) + x^\gamma_k(k, 0)^T P^\gamma x^\gamma_k(k, 0) + x^\gamma_k(k, 0)^T P^h x^\gamma_k(k, 0)
\right.
\]

\[
+ x^\gamma(k, 0)^T P^\gamma x^\gamma(k, 0) - \bar{z}^T(k, 0)\bar{z}(k, 0) + \gamma^2 \omega^T(k, 0)\omega(k, 0) \right\} + E\left\{ x^\gamma(k, 1)^T P^h x^\gamma(k, 1) + x^\gamma(k, 1)^T P^\gamma x^\gamma(k, 1) \right\}
\]

\[
\leq E\left\{ x^\gamma(k, 0)^T P^h x^\gamma(k, 0) + x^\gamma_k(k, 0)^T P^\gamma x^\gamma_k(k, 0) + x^\gamma(k, 0)^T P^\gamma x^\gamma(k, 0) \right\}
\]

\[
+ E\left\{ x^\gamma(k, 1)^T P^h x^\gamma(k, 1) + x^\gamma(k, 1)^T P^\gamma x^\gamma(k, 1) \right\}
\]

\[
+ E\left\{ x^\gamma(0, k + 1)^T P^h x^\gamma(0, k + 1) + x^\gamma(0, k + 1)^T P^\gamma x^\gamma(0, k + 1) \right\}
\]

\[
- E\left\{ \sum_{j=0}^{k} \bar{z}^T(k - j, j)\bar{z}(k - j, j) \right\} + \gamma^2 \sum_{j=0}^{k} \omega^T(k - j, j)\omega(k - j, j),
\]

Adding both sides of the inequality (3.16), we have
and summing such inequalities from \( k = 0 \) to \( k = N \) yields

\[
E \left\{ \sum_{k=0}^{N} \sum_{j=0}^{k} \tilde{z}^T(k-j,j)\tilde{z}(k-j,j) \right\}
\leq \gamma^2 \sum_{k=0}^{N+1} \sum_{j=0}^{k} \omega^T(k-j,j)\omega(k-j,j)
- E \left\{ \sum_{j=0}^{N+1} \tilde{x}(N+1-j,j)P\tilde{x}(N+1-j,j) \right\}
+ E \left\{ \sum_{k=0}^{N+1} \left[ x^h(0,k)^TP^hx^h(0,k) + x^v(k,0)^TP^vx^v(k,0) 
+ x^h_c(0,k)^TP^hx^h_c(0,k) + x^v_c(k,0)^TP^vx^v_c(k,0) \right] \right\}.
\]

Passing to the limit as \( N \to \infty \), we arrive at the inequality

\[
E \left\{ \sum_{k=0}^{\infty} \sum_{j=0}^{k} \tilde{z}^T(k-j,j)\tilde{z}(k-j,j) \right\}
\leq E \left\{ \sum_{k=0}^{\infty} \left[ x^h(0,k)^TP^hx^h(0,k) + x^v(k,0)^TP^vx^v(k,0) 
+ x^h_c(0,k)^TP^hx^h_c(0,k) + x^v_c(k,0)^TP^vx^v_c(k,0) \right] \right\}
+ \gamma^2 \sum_{k=0}^{\infty} \sum_{j=0}^{k} \omega^T(k-j,j)\omega(k-j,j).
\]

For homogeneous initial boundary conditions, the last inequality implies the \( H_\infty \)-performance requirement \( J < 0 \).

3.2. Probability-guaranteed filter design

Theorem 3.1 establishes the mean-square asymptotic stability condition with \( H_\infty \)-disturbance attenuation level \( \gamma \) for the 2D-system (2.6). We proceed with the design of \( H_\infty \)-filters for the system (2.1).

**Theorem 3.2.** Assume that the disturbance attenuation level \( \gamma > 0 \), the probability constraint \( p > 0 \) are given and there are an \( \epsilon > 0 \), positive definite symmetric matrix \( P \) and real-valued...
matrices $W$ and $Q$ such that

$$
\begin{pmatrix}
-P & * & * & * & * & * & * \\
0 & -\varepsilon H_1^TH_1 & * & * & * & * & * \\
0 & 0 & -\gamma^2I & * & * & * & * \\
P\tilde{A}_0^{(l)} + QL_1 & \tilde{a}P & P\tilde{B}_0^{(l)} + QL_2 & -P & * & * & * \\
(I_{n_y} \otimes Q)L_3 & 0 & 0 & 0 & -(I_{n_y} \otimes P) & * & * \\
M_0 - W & 0 & 0 & 0 & 0 & -\varepsilon I & * \\
\varepsilon EH_2 & 0 & 0 & 0 & 0 & 0 & -P
\end{pmatrix} < 0,
$$

where $l = 1, 2, \cdots, 2^q$. Then the filter error system (2.6) is mean-square asymptotically stable and $J < 0$. Moreover, the filter parameters have the form

$$
A_c = N_1P^{-1}QN_2, \quad B_c = N_1P^{-1}QF, \quad C_c = WF,
$$

where

$$
\tilde{A}_0^{(l)} = \text{diag} \{ A^{(l)}, 0 \}, \quad \tilde{B}_0^{(l)} = \begin{bmatrix} B^{(l)} \\ 0 \end{bmatrix}, \quad \tilde{M}_0^{(l)} = \begin{bmatrix} M & 0 \end{bmatrix},
$$

and $A^{(l)}$ and $B^{(l)}$ are $l$-th vertex matrices of the polytope $\Omega$ corresponding to the parameter-box $V_{T_l}$.

Proof. Considering (2.2) and replacing $V_T$ by $V_{T_l}$, shows that (3.5) is valid if

$$
\begin{pmatrix}
-P & * & * & * & * & * & * \\
0 & -\varepsilon H_1^TH_1 & * & * & * & * & * \\
0 & 0 & -\gamma^2I & * & * & * & * \\
P\tilde{A}^{(l)} & \tilde{a}P & P\tilde{B}^{(l)} & -P & * & * & * \\
(I_{n_y} \otimes P)\hat{B} & 0 & 0 & 0 & -(I_{n_y} \otimes P) & * & * \\
\hat{M} & 0 & 0 & 0 & 0 & -\varepsilon I & * \\
\varepsilon EH_2 & 0 & 0 & 0 & 0 & 0 & -P
\end{pmatrix} < 0,
$$

where

$$
\tilde{A}^{(l)} = \begin{bmatrix} A^{(l)} \\ B_c \tilde{C} \end{bmatrix} = \tilde{A}_0^{(l)} + FK_1, \quad \tilde{B}^{(l)} = \begin{bmatrix} B^{(l)} \\ B_c D \end{bmatrix} = \tilde{B}_0^{(l)} + FK_2, \quad l = 1, 2, \cdots, 2^q.
The matrices $\hat{B}$ and $\hat{M}$ can be written in the form $\hat{B} = (I_n \otimes FK)L_3$ and $\hat{M} = \bar{M}_0 - W$, and since for $Q = PK$, the inequality (3.17) follows from (3.18) by straightforward calculations.

**Remark 3.1.** The conditions $A$ and $B$ are connected via parameter-box $V_{T_r}$. Since the vertices $A^{(l)}$ and $B^{(l)}$, $l = 1, 2, \cdots, 2^q$ in (3.17) depend on parameters $a_k$, $b_k$, these parameters should be first determined for all $\theta_k$, $k = 1, \cdots, q$ by Lemma 3.1. This allows to find the vertices of the parameter-box $V_{T_r}$. Note that the probability constraint $p$ is implicitly reflected in the relations (3.2)-(3.4), (3.17).

### 4. Numerical Example

We consider the discrete-time 2D-systems (2.1) with the parameters

$$A(\theta) = A_0 + \theta A_1 = \begin{bmatrix} -0.6 & 0.38 \\ 0.2 & -0.5 \end{bmatrix} + \theta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$B(\theta) = B_0 + \theta B_1 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} + \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.15 & 0.13 \end{bmatrix}. $$

The uncertainty parameter $\theta$ is uniformly distributed over the interval $[-0.05, 0.15]$ and

$$g(x(i,j)) = \begin{bmatrix} 0.2 \sin(x^h(i,j)) \\ -0.1 \sin(x^v(i,j)) \end{bmatrix}. $$

It is easily seen that (2.3) is satisfied with $E = \text{diag} \{0.2, -0.1\}$. The stochastic variable $\alpha(i,j)$ is a Bernoulli distributed white sequence with the expectation $\bar{\alpha} = 0.7$. We also assume the measurement transmission between the plant and the filter is imperfect — i.e. certain data can be lost during the transmission. In the output measurement (2.4), let $\bar{\alpha} = 0.8$, $C = [-1.3, 2]$ and $D = 0.28$. Setting

$$p = 0.9, \quad x(0,0) = \begin{bmatrix} -0.4 \\ -0.82 \end{bmatrix}, \quad x_c(0,0) = \begin{bmatrix} -0.24 \\ 0.02 \end{bmatrix},$$

and using Theorem 3.2, we obtain the associated filter matrices

$$A_c = \begin{bmatrix} -0.0998 & 0.1485 \\ 0.0265 & -0.0385 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0.0923 \\ -0.0242 \end{bmatrix},$$

$$C_c = \begin{bmatrix} 4.086 \times 10^{-16} & -3.085 \times 10^{-15} \end{bmatrix}. $$

To show the stability of the filtering error system we assume that $\sigma(i,j) = 0$. Figs. 1 and 2 show the state response of $x^h(i,j)$ and $x^v(i,j)$. The corresponding state estimates $x^h_c(i,j)$ and $x^v_c(i,j)$ are presented in Figs. 3 and 4, whereas the filtering error signal $\tilde{z}(i,j)$ is shown in Fig. 5. To demonstrate the performance of the constructed filter, we choose external disturbance $\sigma(i,j) = 0.05 \sin(10i,j)$ and display the response of the filtering error signal $\tilde{z}(i,j)$ in Fig. 6. Note that under the condition $\sigma(i,j) = 0$, the filter error system is mean-square asymptotically stable and the $H_\infty$-performance is satisfied.
Figure 1: $\omega(i,j) = 0$. The state $x(i,j)$.

Figure 2: $\omega(i,j) = 0$. The state $x^*(i,j)$.

Figure 3: $\omega(i,j) = 0$. The state estimate $x^h(i,j)$. 
Figure 4: \( \omega(i, j) = 0 \). The state estimate \( x^*_c(i, j) \).

Figure 5: The estimate error \( \hat{z}(i, j) \) with \( \omega(i, j) = 0 \).

Figure 6: The estimate error \( \hat{z}(i, j) \) with \( \omega(i, j) \neq 0 \).
5. Conclusions

We studied a probability-guaranteed $H_\infty$-filtering problem for 2D-discrete time systems with random nonlinearities and missing measurements. To characterise the statistical uncertain parameters, the uniform distribution is used. The filter parameters are derived by LMI technique and we use a special parameter-box to guarantee an $H_\infty$-performance with pre-specified probability constraints. An example shows the effectiveness of the filtering scheme developed.

Acknowledgments

This work was supported by the Heilongjiang Postdoctoral Scientific Research Developmental Fund of China under Grant LBH-Q 16121.

References