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The Decoupling of Elastic Waves from a Weak Formulation Perspective

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Abstract. Two weak formulations for the Lamé system with the boundary conditions of third and fourth types are proposed. It is shown that the regularity of the solutions and properties of the boundary surface guarantee the equivalence of variational and standard formulations of the problem. Moreover, if the boundary of Ω is a Lipschitz polyhedron or if $\mathscr{S}(\boldsymbol{x}) = 0$ on $\partial\Omega$, the decoupling results of [8] are derived from the weak formulations.

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply-connected domain with a connected piecewise $C^{2,1}$-smooth boundary $\partial\Omega$. An isotropic elastic medium characterised by the Lamé constants $\lambda > 0$ and $\mu > 0$, occupies Ω. Given a source term $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, we consider the linearised elasticity equation

$$-\Delta^* \boldsymbol{u} - \omega^2 \boldsymbol{u} = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{1.1}$$

where $\boldsymbol{u} = [u_j(\boldsymbol{x})]_{j=1}^3$ is the displacement field, $\omega > 0$ the angular wavenumber, and the operator Δ^* is defined by

$$\Delta^* \boldsymbol{u} := \mu \Delta \boldsymbol{u} + (\lambda + \mu) \nabla(\nabla \cdot \boldsymbol{u}) = -\mu \nabla \wedge (\nabla \wedge \boldsymbol{u}) + (\lambda + 2\mu) \nabla(\nabla \cdot \boldsymbol{u}), \tag{1.2}$$

where $\wedge$ is the cross product of two vector fields.

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The Lamé system (1.1) is complemented by the boundary conditions of different kinds [6,7]. The first kind boundary condition has the form

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega.$$

The second one reads

$$\boldsymbol{Tu} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega,$$

with the traction operator **T**,

$$\boldsymbol{Tu} := \lambda(\nabla\cdot\boldsymbol{u})\boldsymbol{\nu} + \mu(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)\boldsymbol{\nu} = 2\mu\partial_{\boldsymbol{\nu}}\boldsymbol{u} + \lambda\,\boldsymbol{\nu}\nabla\cdot\boldsymbol{u} + \mu\,\boldsymbol{\nu}\wedge(\nabla\wedge\boldsymbol{u}),$$

where T is the transposition operation, $\boldsymbol{\nu}\in\mathbb{S}^2$ the outward unit normal vector to $\partial\Omega$, and $\partial_{\boldsymbol{\nu}}$ a boundary differential operator defined by

$$\partial_{\boldsymbol{\nu}}\boldsymbol{u} := [\,\boldsymbol{\nu}\cdot\nabla u_j\,]_{j=1}^3.$$

The equations

$$\boldsymbol{\nu}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{\nu}\wedge\boldsymbol{Tu} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

and

$$\boldsymbol{\nu}\wedge\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{\nu}\cdot\boldsymbol{Tu} = 0 \quad \text{on} \quad \partial\Omega \tag{1.4}$$

represent the boundary conditions of the third and fourth kind, respectively.

Here, we mainly focus on the Lamé system (1.1) provided with the boundary condition (1.3) or (1.4). It is known that the elastic body waves can be decomposed into two parts — viz. pressure and shear waves. These waves generally coexist and simultaneously propagate with different speeds and in different directions. In recent work [8], a complete decoupling of the pressure and shear waves is established for the third and fourth boundary conditions under geometric conditions on the boundary of an impenetrable scatterer. This allows to reformulate the Lamé system as Helmholtz and Maxwell systems.

The goal of this work is to revisit the decoupling results [8] in order to obtain new relevant variational formulations and to show the well-posedness of the corresponding variational problems. We observe that the problems (3.5) and (3.11) are well-posed without any geometric assumption concerning the boundary surface. Furthermore, under an assumption about the regularity of the solution, the variational formulations are rewritten as Helmholtz and Maxwell systems. If the boundary satisfies additional geometric conditions, the variational formulations and the original Lamé system are equivalent. This means that the decoupling results deduced from the Lamé system in [8] are retrieved from a weak formulation perspective.

To the best of our knowledge, the variational formulations (3.5) and (3.11) are novel and they can find numerous applications in numerical simulation of the Lamé system with the boundary conditions (1.3) and (1.4), including the variational discretisation such as finite element methods [2], [3], [5], [9]. Due to Theorems 3.3 and 3.6, computations and the related numerical analysis may be performed separately on the Helmholtz equation and the Maxwell system.

The rest of this paper is organised as follows. Auxiliary results needed for the analysis of the methods are provided in Section 2. In Section 3, we present variational formulations for the Lamé system (1.1) associated with the third (1.3) and fourth (1.4) boundary conditions and derive the relevant decoupling results. Section 4 contains our concluding remarks.

2. Preliminaries

Let us start with L^2-based Sobolev spaces. By $(\cdot,\cdot)_G$ we denote the $L^2(G)$ scalar product on an open bounded domain $G \subset \mathbb{R}$ and we write $(\cdot,\cdot)$ if $G = \Omega$. The spaces $\boldsymbol{H}(\mathbf{curl};\Omega)$ and $\boldsymbol{H}(\mathrm{div};\Omega)$ are defined by — cf. [1]:

$$\begin{aligned}\boldsymbol{H}(\mathbf{curl};\Omega) &= \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{\nabla} \wedge \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\right\},\\ \boldsymbol{H}(\mathrm{div};\Omega) &= \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{\nabla} \cdot \boldsymbol{v} \in L^2(\Omega)\right\},\end{aligned}$$

and equipped with the norms

$$\begin{aligned}\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} &= \left(\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{\nabla} \wedge \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}\right)^{1/2},\\ \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div};\Omega)} &= \left(\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{\nabla} \cdot \boldsymbol{v}\|^2_{L^2(\Omega)}\right)^{1/2}.\end{aligned}$$

Let us also introduce the space

$$\boldsymbol{X} := \boldsymbol{H}(\mathbf{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div};\Omega)$$

with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{X}} = \left(\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{\nabla} \wedge \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{\nabla} \cdot \boldsymbol{v}\|^2_{L^2(\Omega)}\right)^{1/2}.$$

The main solution spaces considered in this work are the following subspaces of $\boldsymbol{X}$:

$$\boldsymbol{X}_N := \{\boldsymbol{v} \in \boldsymbol{X};\, \boldsymbol{\nu} \wedge \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \partial\Omega\}, \quad \boldsymbol{X}_T := \{\boldsymbol{v} \in \boldsymbol{X};\, \boldsymbol{\nu} \cdot \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \partial\Omega\}.$$

Lemma 2.1 (cf. Weber [11])**.** *If Ω is a bounded connected Lipschitz domain, then the spaces $\boldsymbol{X}_N$ and $\boldsymbol{X}_T$ are compactly embedded into $\boldsymbol{L}^2(\Omega)$.*

Let Γ be the regular part of the boundary $\partial\Omega$ with the parametric representation

$$\boldsymbol{x}(\boldsymbol{u}) = (x_1(u_1,u_2), x_2(u_1,u_2), x_3(u_1,u_2))^{\mathrm{T}}, \quad \boldsymbol{u} = (u_1,u_2)^{\mathrm{T}} \in \mathbb{R}^2, \tag{2.1}$$

and let $g = (g_{jk})^2_{j,k=1}$ be the first fundamental matrix of differential geometry for Γ such that

$$g_{jk} := \frac{\partial \boldsymbol{x}}{\partial u_j} \cdot \frac{\partial \boldsymbol{x}}{\partial u_k}, \quad j,k = 1,2.$$

Here and in what follows, we assume that the parameterisation (2.1) is chosen so that

$$\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{x}) = \frac{1}{\sqrt{|g|}} \frac{\partial \boldsymbol{x}}{\partial u_1} \wedge \frac{\partial \boldsymbol{x}}{\partial u_2}, \quad |g| := \det(g),$$

is the outward unit normal vector to $\partial\Omega$ on Γ. We denote by Grad_Γ the surface gradient operator on Γ — cf. [4, 10]. If φ is a sufficiently smooth function defined in an open neighborhood of Γ, then

$$\mathrm{Grad}_\Gamma \varphi = \sum_{j,k=1}^{2} g^{jk} \frac{\partial \varphi}{\partial u_j} \frac{\partial \boldsymbol{x}}{\partial u_k},$$

where

$$\left(g^{jk}\right)_{j,k=1}^{2} := \left[\left(g_{jk}\right)_{j,k=1}^{2}\right]^{-1}.$$

We also consider an operator $\mathscr{S}$ defined by

$$\mathscr{S}(\boldsymbol{x}) = \mathscr{S}_\Gamma(\boldsymbol{x}) := \frac{1}{2} \sum_{l=1}^{3} \left[\mathrm{Grad}_\Gamma \nu_l\right]_l.$$

3. Weak Formulations and Decoupling

Let us present the main results of this paper. The application of the Helmholtz decomposition to the source $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ yields $\boldsymbol{f} = \boldsymbol{f}_p + \boldsymbol{f}_s$ with $\nabla \wedge \boldsymbol{f}_p = \boldsymbol{0}$ and $\nabla \cdot \boldsymbol{f}_s = 0$. Hence, the solution $\boldsymbol{u}$ of (1.1) can be formally written in the form

$$\boldsymbol{u} = \boldsymbol{u}_p + \boldsymbol{u}_s,$$

where

$$\begin{aligned} \boldsymbol{u}_p &:= -\frac{1}{k_p^2}\left(\nabla(\nabla \cdot \boldsymbol{u}) + \frac{1}{\lambda + 2\mu} \boldsymbol{f}_p\right), \\ \boldsymbol{u}_s &= \frac{1}{k_s^2}\left(\nabla \wedge (\nabla \wedge \boldsymbol{u}) - \frac{1}{\mu} \boldsymbol{f}_s\right), \end{aligned}$$

and $k_p := \omega/\sqrt{\lambda + 2\mu}$, $k_s := \omega/\sqrt{\mu}$. The vector fields $\boldsymbol{u}_p$ and $\boldsymbol{u}_s$ are referred to as the pressure (longitudinal) and shear (transversal) parts of $\boldsymbol{u}$, respectively. It is easily seen that

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{u}_p, \quad \nabla \wedge \boldsymbol{u} = \nabla \wedge \boldsymbol{u}_s.$$

Now we are going to derive a decomposition of $\boldsymbol{u}$ by using suitable Helmholtz equation and Maxwell system with appropriately chosen boundary conditions for $\nabla \cdot \boldsymbol{u}$ and $\nabla \wedge \boldsymbol{u}$, respectively.

3.1. Boundary conditions (1.4)

In order to introduce the weak formulation, we recall a decoupling result for the elastic scattering problem governed by the Lamé system (1.1) with the zero source. Note that the corresponding proof does not involve the information concerning the right-hand side of (1.1).

Lemma 3.1 (cf. Liu & Xiao [8]). *Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ be the solution of the corresponding Eq.* (1.1). *If $\mathscr{S}(\boldsymbol{x}) = 0$ on the connected part Γ of $\partial\Omega$, then the boundary condition* (1.4) *yields*

$$\nabla \cdot \boldsymbol{u} = 0 \quad on \quad \Gamma. \tag{3.1}$$

Suppose that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ is the solution of the Eq. (1.1) with the boundary condition (1.4). Integrating by parts, for any $\boldsymbol{v} \in \boldsymbol{X}_N$ we get

$$(\nabla \wedge \nabla \wedge \boldsymbol{u}, \boldsymbol{v}) = (\nabla \wedge \boldsymbol{u}, \nabla \wedge \boldsymbol{v}), \tag{3.2}$$

$$-(\nabla(\nabla \cdot \boldsymbol{u}), \boldsymbol{v}) = (\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}) - \langle \nabla \cdot \boldsymbol{u}, \boldsymbol{v} \cdot \boldsymbol{\nu} \rangle_{\partial\Omega}, \tag{3.3}$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ is the duality pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ since the normal trace space of $\boldsymbol{H}(\mathrm{div};\Omega)$ is $H^{-1/2}(\partial\Omega)$ — cf. [9]. Assume now that $\partial\Omega = \cup_{j=1}^m \Gamma_j$ and $\mathscr{S}(\boldsymbol{x}) = 0$ on each connected component Γ_j. The Eqs. (3.2), (3.3), (3.1) then yield

$$\begin{aligned}(-\Delta^* \boldsymbol{u}, \boldsymbol{v}) &= (\mu \nabla \wedge \nabla \wedge \boldsymbol{u}, \boldsymbol{v}) - ((\lambda + 2\mu)\nabla(\nabla \cdot \boldsymbol{u}), \boldsymbol{v}) \\ &= (\mu \nabla \wedge \boldsymbol{u}, \nabla \wedge \boldsymbol{v}) + ((\lambda + 2\mu)\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}). \end{aligned} \tag{3.4}$$

It follows from (3.4) and (1.1) that the solution $\boldsymbol{u}$ satisfies the equation

$$(\mu \nabla \wedge \boldsymbol{u}, \nabla \wedge \boldsymbol{v}) + ((\lambda + 2\mu)\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}) - \omega^2(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N.$$

The variational formulation of the Lamé problem (1.1) with the boundary condition (1.4) can be written as:

Provided that $\mathscr{S}(\boldsymbol{x}) = 0$ on $\partial\Omega$, find $\boldsymbol{u} = \boldsymbol{u}([\Omega, \mathrm{IV}]) \in \boldsymbol{X}_N$ such that

$$a_4(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N, \tag{3.5}$$

where $a_4(\cdot, \cdot)$ is the bilinear form defined by

$$a_4(\boldsymbol{v}, \boldsymbol{w}) := (\mu \nabla \wedge \boldsymbol{v}, \nabla \wedge \boldsymbol{w}) + ((\lambda + 2\mu)\nabla \cdot \boldsymbol{v}, \nabla \cdot \boldsymbol{w}) - \omega^2(\boldsymbol{v}, \boldsymbol{w}), \quad \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}_N.$$

Theorem 3.1. *If $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, then for almost all $\omega > 0$ the Eq.* (3.5) *has a unique solution $\boldsymbol{u} \in \boldsymbol{X}_N$ such that*

$$\|\boldsymbol{u}\|_X \le C\|\boldsymbol{f}\|_{L^2(\Omega)}, \tag{3.6}$$

where C is a constant independent of $\boldsymbol{f}$.

Proof. For any $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$, we define an operator $\boldsymbol{K}_N : \boldsymbol{L}^2(\Omega) \to \boldsymbol{X}_N$ by

$$a_4(\boldsymbol{K}_N \boldsymbol{w}, \boldsymbol{v}) + (1 + \omega^2)(\boldsymbol{K}_N \boldsymbol{w}, \boldsymbol{v}) = (\boldsymbol{w}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N.$$

With respect to the graph norm, one can easily check that $a_4(\cdot, \cdot) + (1 + \omega^2)(\cdot, \cdot)$ is a bounded and coercive form on $\boldsymbol{X}_N$. By the Lax-Milgram lemma, the operator $\boldsymbol{K}_N$ is well-defined and the inequality

$$\|\boldsymbol{K}_N \boldsymbol{w}\|_X \le C_1 \|\boldsymbol{w}\|_{L^2(\Omega)}$$

holds. Hence $\boldsymbol{K}_N : \boldsymbol{L}^2(\Omega) \to \boldsymbol{X}_N$ is a bounded linear operator and we can rewrite (3.5) as the problem of finding of $\boldsymbol{u} \in \boldsymbol{X}_N$ such that

$$\boldsymbol{u} - \left(\omega^2 + 1\right) \boldsymbol{K}_N \boldsymbol{u} = \boldsymbol{K}_N \boldsymbol{f}.$$

Lemma 2.1 implies that $\boldsymbol{X}_N$ is compactly embedded into $\boldsymbol{L}^2(\Omega)$, so that $\boldsymbol{K}_N : \boldsymbol{X}_N \to \boldsymbol{X}_N$ is compact. Since compact operators have at most countable number of eigenvalues, the Eq. (3.5) has unique solution for almost all $\omega > 0$ and for such ω the estimate (3.6) follows. □

Remark 3.1. From the perspective of variational problems, Theorem 3.1 means that (3.5) itself is a well-posed forward problem without any additional geometric conditions $\mathscr{S}(\boldsymbol{x}) = 0$ on $\partial\Omega$.

The following theorem retrieves the boundary condition in (3.1) from the variational formulation (3.5).

Theorem 3.2. *If the solution $\boldsymbol{u} = \boldsymbol{u}[\Omega; IV]$ of the problem* (3.5) *belongs to the space $\boldsymbol{H}^2(\Omega)$, then it is also the solution of the problem* (1.1) *with the boundary conditions*

$$\boldsymbol{\nu} \wedge \boldsymbol{u} = \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{u} = 0 \quad \textit{on} \quad \partial\Omega. \tag{3.7}$$

Proof. To show that the solution to (3.5) is also a solution of (1.1), one can use integration by parts and (1.2). The first boundary condition in (3.7) follows directly from the definition of $\boldsymbol{X}_N$. According to the trace theorem for Sobolev spaces [9], on any regular component Γ of $\partial\Omega$ and for any $\phi \in C_c^\infty(\Gamma)$, there exists a $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that

$$\boldsymbol{\nu} \wedge \boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega, \quad \boldsymbol{v} \cdot \boldsymbol{\nu} = \begin{cases} \phi & \text{on} \quad \Gamma, \\ 0 & \text{on the rest of} \quad \partial\Omega. \end{cases}$$

It is clear that such a vector $\boldsymbol{v}$ belongs to $\boldsymbol{X}_N$. Substituting it into (3.5), integrating by parts and noting (1.1), we obtain that

$$((\lambda + 2\mu)\nabla \cdot \boldsymbol{u}, \phi)_\Gamma = 0 \quad \forall \phi \in C_c^\infty(\Gamma).$$

Since Γ is an arbitrary component, the second equation in (3.7) follows. □

Remark 3.2. Imposing the condition $\mathscr{S}(\boldsymbol{x}) = 0$ on $\partial\Omega$ and using two boundary conditions in (3.7) along with the arguments in the proof of [8, Theorem 2.1], we arrive at the second boundary condition in (1.4). This shows that (3.5) is equivalent to (1.1) and (1.4) if $\mathscr{S}(\boldsymbol{x}) = 0$ on $\partial\Omega$.

The weak formulation (3.5) can be now used to establish the following decoupling result.

Theorem 3.3. *If the solution $\boldsymbol{u} = \boldsymbol{u}([\Omega,IV])$ of the problem* (3.5) *belongs to $\boldsymbol{H}^3(\Omega)$ and $\boldsymbol{f} \in \boldsymbol{H}^1(\Omega)$, then $v_p = v_p([\Omega,IV]) =: -\nabla\cdot\boldsymbol{u}$ is the solution of the Helmholtz system*

$$\begin{aligned} &\Delta v_p + k_p^2 v_p = \frac{1}{\lambda+2\mu}\nabla\cdot\boldsymbol{f} \quad in \quad \Omega,\\ &v_p = 0 \quad on \quad \partial\Omega, \end{aligned} \tag{3.8}$$

and $\boldsymbol{E}_s = \boldsymbol{E}_s([\Omega,IV]) := \nabla\wedge\boldsymbol{u}$ the solution of the Maxwell system

$$\begin{aligned} &\nabla\wedge(\nabla\wedge\boldsymbol{E}_s) - k_s^2\boldsymbol{E}_s = \frac{1}{\mu}\nabla\wedge\boldsymbol{f} \quad in \quad \Omega,\\ &\nabla\cdot\boldsymbol{E}_s = 0 \quad in \quad \Omega,\\ &\boldsymbol{\nu}\wedge(\nabla\wedge\boldsymbol{E}_s) = \frac{1}{\mu}\boldsymbol{\nu}\wedge\boldsymbol{f} \quad on \quad \partial\Omega. \end{aligned} \tag{3.9}$$

Proof. If $\phi \in C_c^\infty(\Omega)$, integration by parts yields

$$\begin{aligned} a_4(\boldsymbol{u},\nabla\phi) &= ((\lambda+2\mu)\nabla\cdot\boldsymbol{u},\nabla\cdot\nabla\phi) - \omega^2(\boldsymbol{u},\nabla\phi)\\ &= -((\lambda+2\mu)\nabla(\nabla\cdot\boldsymbol{u}),\nabla\phi) + \omega^2(\nabla\cdot\boldsymbol{u},\phi). \end{aligned}$$

Moreover, we also have

$$(\boldsymbol{f},\nabla\phi) = -(\nabla\cdot\boldsymbol{f},\phi).$$

Therefore, using (3.5) and the second equation in (3.7), we arrive at the Helmholtz system (3.8) with $v_p = -\nabla\cdot\boldsymbol{u}$. To derive the system (3.9), we set $\boldsymbol{v} := \nabla\wedge\boldsymbol{F}$, $\boldsymbol{F} \in \boldsymbol{C}_c^\infty(\Omega)$ and substitute it in (3.5), so that

$$\begin{aligned} a_4(\boldsymbol{u},\nabla\wedge\boldsymbol{F}) &= (\mu\nabla\wedge\boldsymbol{u},\nabla\wedge\nabla\wedge\boldsymbol{F}) - \omega^2(\boldsymbol{u},\nabla\wedge\boldsymbol{F})\\ &= (\mu\nabla\wedge\nabla\wedge(\nabla\wedge\boldsymbol{u}),\boldsymbol{F}) - \omega^2(\nabla\wedge\boldsymbol{u},\boldsymbol{F}), \end{aligned}$$

and

$$(\boldsymbol{f},\nabla\wedge\boldsymbol{F}) = (\nabla\wedge\boldsymbol{f},\boldsymbol{F}).$$

Thus the first and second equations in (3.9) are obtained with $\boldsymbol{E}_s = \nabla\wedge\boldsymbol{u}$. To derive the third boundary condition, we invoke Theorem 3.2. Since $\boldsymbol{u}$ is the solution of (1.1), then

$$\mu\boldsymbol{\nu}\wedge\nabla\wedge(\nabla\wedge\boldsymbol{u}) - (\lambda+2\mu)\boldsymbol{\nu}\wedge\nabla(\nabla\cdot\boldsymbol{u}) - \omega^2\boldsymbol{\nu}\wedge\boldsymbol{u} = \boldsymbol{\nu}\wedge\boldsymbol{f} \quad \text{on} \quad \partial\Omega,$$

and taking into account the relations (3.7), we finish the proof. □

Remark 3.3. During the proof of Theorem 3.3, we also derived the boundary condition

$$\boldsymbol{\nu}\wedge\nabla\wedge(\nabla\wedge\boldsymbol{u}) = \frac{1}{\mu}\boldsymbol{\nu}\wedge\boldsymbol{f} \quad \text{on} \quad \partial\Omega.$$

This and the second boundary condition in (3.7) are deduced from (1.1) and (1.4) with $\boldsymbol{f} = \boldsymbol{0}$ under the assumption that $\mathscr{S}(\boldsymbol{x}) = 0$ on $\partial\Omega$ in [8]. The decoupling (3.8) and (3.9) is then established. Here we retrieve these results without any geometric condition — cf. also the proof of Theorem 3.2. There is no conflict here, since the variational formulation (3.5) is derived from (1.1) and (1.4) based on the geometric assumption. In other words, (3.5) is independent of $\mathscr{S}(\boldsymbol{x})$ due to the geometric assumption $\mathscr{S}(\boldsymbol{x}) = 0$ on $\partial\Omega$.

3.2. Boundary Condition (1.3)

Let us now consider the Eq. (1.1) with the boundary condition (1.3). To proceed, we need an auxiliary result.

Lemma 3.2 (cf. Liu & Xiao [8]). *Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{u}$ be the corresponding solution of* (1.1). *If the connected component Γ of $\partial\Omega$ is flat, then the third kind boundary condition* (1.3) *yields*

$$\boldsymbol{\nu} \wedge (\boldsymbol{\nabla} \wedge \boldsymbol{u}) = \boldsymbol{0} \quad on \quad \Gamma. \tag{3.10}$$

Suppose $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ is the solution of (1.1) and (1.3). Using again integration by parts, for any $\boldsymbol{v} \in \boldsymbol{X}_T$ we get

$$\begin{aligned} -(\Delta^* \boldsymbol{u}, \boldsymbol{v}) =& (\mu \boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \wedge \boldsymbol{u}, \boldsymbol{v}) - ((\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}), \boldsymbol{v}) \\ =& (\mu \boldsymbol{\nabla} \wedge \boldsymbol{u}, \boldsymbol{\nabla} \wedge \boldsymbol{v}) + ((\lambda + 2\mu)\boldsymbol{\nabla} \cdot \boldsymbol{u}, \boldsymbol{\nabla} \cdot \boldsymbol{v}) \\ &+ \langle \mu \boldsymbol{\nu} \wedge (\boldsymbol{\nabla} \wedge \boldsymbol{u}), \boldsymbol{v} \rangle_{\partial\Omega}. \end{aligned}$$

If, in addition, Ω is a Lipschitz polyhedron, then employing (3.10), (1.1), we obtain that $\boldsymbol{u}$ satisfies the equation

$$(\mu \boldsymbol{\nabla} \wedge \boldsymbol{u}, \boldsymbol{\nabla} \wedge \boldsymbol{v}) + ((\lambda + 2\mu)\boldsymbol{\nabla} \cdot \boldsymbol{u}, \boldsymbol{\nabla} \cdot \boldsymbol{v}) - \omega^2(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}_T.$$

Thus for Lipschitz polyhedrons Ω, the boundary value problem (1.1) with the third kind boundary condition (1.3) can be formulated as follows:

Find $\boldsymbol{u} = \boldsymbol{u}[\Omega; III] \in \boldsymbol{X}_T$ such that

$$a_3(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}_T, \tag{3.11}$$

where

$$a_3(\boldsymbol{v}, \boldsymbol{w}) := (\mu \boldsymbol{\nabla} \wedge \boldsymbol{v}, \boldsymbol{\nabla} \wedge \boldsymbol{w}) + ((\lambda + 2\mu)\boldsymbol{\nabla} \cdot \boldsymbol{v}, \boldsymbol{\nabla} \cdot \boldsymbol{w}) - \omega^2(\boldsymbol{v}, \boldsymbol{w}), \quad \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}_T.$$

Theorem 3.4. *If $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, then for almost all $\omega > 0$ the Eq.* (3.11) *has a unique solution $\boldsymbol{u} \in \boldsymbol{X}_T$ such that*

$$\|\boldsymbol{u}\|_X \le C \|\boldsymbol{f}\|_{L^2(\Omega)}, \tag{3.12}$$

where C is a constant independent of $\boldsymbol{f}$.

Proof. Similar to the proof of Theorem 3.1, we rewrite the Eq. (3.11) as

$$\boldsymbol{u} - (\omega^2 + 1)\boldsymbol{K}_T \boldsymbol{u} = \boldsymbol{K}_T \boldsymbol{f}$$

with the operator $\boldsymbol{K}_T : \boldsymbol{L}^2(\Omega) \to \boldsymbol{X}_T$ defined by

$$a_3(\boldsymbol{K}_T \boldsymbol{w}, \boldsymbol{v}) + (1 + \omega^2)(\boldsymbol{K}_T \boldsymbol{w}, \boldsymbol{v}) = (\boldsymbol{w}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}_T.$$

By Lemma 2.1, $\boldsymbol{X}_T$ is compactly embedded into $\boldsymbol{L}^2(\Omega)$. Therefore, the theory of compact operators yields the well-posedness of the problem (3.11) and (3.12) — cf. the proof of Theorem 3.1. □

Theorem 3.5. *If the solution* $\boldsymbol{u} = \boldsymbol{u}[\Omega; III]$ *of the Eq.* (3.11) *belongs to* $\boldsymbol{H}^2(\Omega)$*, then it is also the solution of the Eq.* (1.1) *with the boundary conditions*

$$\boldsymbol{\nu} \cdot \boldsymbol{u} = 0, \quad \boldsymbol{\nu} \wedge (\boldsymbol{\nabla} \wedge \boldsymbol{u}) = \boldsymbol{0} \quad on \quad \partial\Omega. \tag{3.13}$$

Proof. Integration by parts shows that $\boldsymbol{u}$ satisfies the Eq. (1.1) and since $\boldsymbol{u} \in \boldsymbol{X}_T$, it has a vanishing normal trace on $\partial\Omega$ — i.e. it also satisfies the first condition in (3.13). For the second condition in (3.13), we observe that for any tangential vector field $\boldsymbol{\phi} \in (C_c^\infty(\Gamma))^2$ considered on regular piece Γ of $\partial\Omega$, the Trace Theorem for Sobolev spaces provides the existence of a vector $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that

$$\begin{aligned} \boldsymbol{\nu} \wedge \boldsymbol{v} &= \begin{cases} \boldsymbol{\phi} & \text{on} \quad \Gamma, \\ 0 & \text{on the rest of} \quad \partial\Omega, \end{cases} \\ \boldsymbol{\nu} \cdot \boldsymbol{v} &= \boldsymbol{0} \quad \text{on} \quad \partial\Omega. \end{aligned}$$

The vector $\boldsymbol{v}$ obviously belongs to $\boldsymbol{X}_T$. Substituting it into (3.11), integrating the result by parts and using (1.1), we arrive at the equation

$$-(\mu \boldsymbol{\nu} \wedge (\boldsymbol{\nabla} \wedge \boldsymbol{u}), \boldsymbol{v})_{\partial\Omega} = (\mu \boldsymbol{\nu} \wedge (\boldsymbol{\nabla} \wedge \boldsymbol{u}), \boldsymbol{\nu} \wedge \boldsymbol{\phi})_\Gamma = 0.$$

Since Γ is an arbitrary connected component of $\partial\Omega$, the second boundary condition follows and the proof is completed. □

Remark 3.4. Theorem 3.4 means that (3.11) is a well-posed problem even without the assumption that Ω is a Lipschitz polyhedron.

Remark 3.5. Similar to the case of boundary conditions of fourth type, we can show that the assumption that Ω is a Lipschitz polyhedron guarantees that (3.13) implies the second condition in (1.3). Therefore, the corresponding variational problem (3.11) and the Lamé system (1.1) with (1.3) are equivalent.

Theorem 3.6. *If* $\boldsymbol{f} \in \boldsymbol{H}^1(\Omega)$ *and the solution* $\boldsymbol{u} = \boldsymbol{u}([\Omega, III])$ *of* (3.11) *belongs to* $\boldsymbol{H}^3(\Omega)$*, then*

$$\begin{aligned} \Delta v_p + k_p^2 v_p &= \frac{1}{\lambda + 2\mu} \boldsymbol{\nabla} \cdot \boldsymbol{f} \quad in \quad \Omega, \\ \partial_\nu v_p &= \frac{1}{\lambda + 2\mu} \boldsymbol{\nu} \cdot \boldsymbol{f} \quad on \quad \partial\Omega, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \boldsymbol{E}_s) - k_s^2 \boldsymbol{E}_s = \frac{1}{\mu} \boldsymbol{\nabla} \wedge \boldsymbol{f} \quad in \quad \Omega, \\ &\boldsymbol{\nabla} \cdot \boldsymbol{E}_s = 0 \quad in \quad \Omega, \\ &\boldsymbol{\nu} \wedge \boldsymbol{E}_s = \boldsymbol{0} \quad on \quad \partial\Omega, \end{aligned} \tag{3.15}$$

where $v_p = v_p([\Omega, III]) =: -\boldsymbol{\nabla} \cdot \boldsymbol{u}$ *and* $\boldsymbol{E}_s = \boldsymbol{E}_s([\Omega, III]) := \boldsymbol{\nabla} \wedge \boldsymbol{u}$.

Proof. The proof is similar to that of Theorem 3.3. We start with the Maxwell system (3.15). Let $\boldsymbol{F} \in \boldsymbol{C}_c^{\infty}(\Omega)$. Substituting $\boldsymbol{v} = \nabla \wedge \boldsymbol{F}$ in (3.11) and integrating the result by parts yields

$$(\mu\nabla \wedge \nabla \wedge (\nabla \wedge \boldsymbol{u}), \boldsymbol{F}) - \omega^2(\nabla \wedge \boldsymbol{u}, \boldsymbol{F}) = (\nabla \wedge \boldsymbol{f}, \boldsymbol{F}).$$

This and the second equation in (3.13) show that $\boldsymbol{E}_s = \nabla \wedge \boldsymbol{u}$ satisfies (3.15). Analogously, using the vector $\boldsymbol{v} = \nabla\phi$, $\phi \in C_c^{\infty}(\Omega)$ in (3.11), one obtains that $v_p = -\nabla \cdot \boldsymbol{u}$ satisfies the first equation in (3.14). To show that this vector also satisfy the boundary condition in (3.14), we recall that $\boldsymbol{u}$ is a solution of the Lamé system (1.1) by Theorem 3.5. Choosing an arbitrary $\phi \in H^1(\Omega)$, we multiply both sides of (1.1) by $\nabla\phi$. Then for any $\phi \in H^1(\Omega)$ integration by parts and (1.2) yield

$$\begin{aligned}
&(-\Delta^* \boldsymbol{u} - \omega^2 \boldsymbol{u}, \nabla\phi) = (\mu\nabla \wedge \nabla \wedge \boldsymbol{u}, \nabla\phi) - ((\lambda + 2\mu)\nabla(\nabla \cdot \boldsymbol{u}), \nabla\phi) - \omega^2(\boldsymbol{u}, \nabla\phi) \\
&\underbrace{=}_{(3.13)} ((\lambda + 2\mu)\Delta\nabla \cdot \boldsymbol{u}, \phi) - ((\lambda + 2\mu)\nabla(\nabla \cdot \boldsymbol{u}) \cdot \boldsymbol{v}, \phi)_{\partial\Omega} + \omega^2(\nabla \cdot \boldsymbol{u}, \phi),
\end{aligned}$$

$$(\boldsymbol{f}, \nabla\phi) = -(\nabla \cdot \boldsymbol{f}, \phi) + (\boldsymbol{v} \cdot \boldsymbol{f}, \phi)_{\partial\Omega} \quad \forall\phi \in H^1(\Omega),$$

and to finish the proof, one has to use (1.1) and the first equation in (3.14). □

Remark 3.6. In the proof above, another boundary condition – viz.

$$\nabla \cdot \nabla(\nabla \cdot \boldsymbol{u}) = -\frac{1}{\lambda + 2\mu}\nabla \cdot \boldsymbol{f} \quad \text{on} \quad \partial\Omega \tag{3.16}$$

is obtained. Along with the second condition in (3.13), condition (3.16) has been derived in [8] for the homogeneous Lamé system coupled with (1.3) under the assumption that Ω is a Lipschitz polyhedron. This ensures the validity of (3.14) and (3.15). These results are recalled here since (3.11) is related to Lipschitz polyhedrons.

4. Concluding Remarks

We propose two weak formulations for the Lamé system with the boundary conditions of third and fourth types. The regularity of the solutions and properties of the boundary surface guaranty the equivalence of variational and standard formulations of the problem. Moreover, if the boundary Ω is a Lipschitz polyhedron or if $\mathscr{S}(\boldsymbol{x}) = 0$ on $\partial\Omega$, the decoupling results of [8] are derived from the weak formulations. The validity of these results without of assumptions mentioned are subject to additional studies.

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