East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.220418.210718

Vol. **8**, No. 4, pp. 746-763
November 2018

Convergence of Parareal Algorithms for PDEs with Fractional Laplacian and a Non-Constant Coefficient

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Received 22 April 2018; Accepted (in revised version) 21 July 2018.

Abstract. The convergence of Parareal-Euler and -LIIIC2 algorithms using the backward Euler method as a $\mathscr{G}$-propagator for the linear problem $\mathbf{U}'(t)+\alpha(t)\mathbf{A}^{\eta}\mathbf{U}(t)=\mathbf{f}(t)$ with a non-constant coefficient α is studied. We propose to employ the propagator G to a constant model $\mathbf{U}'(t)+\beta\mathbf{A}^{\eta}\mathbf{U}(t)=\mathbf{f}(t)$ with a special coefficient β instead of applying both propagators $\mathscr{G}$ and $\mathscr{F}$ to the same target model. We established a simple formula to find an optimal parameter β_{opt}, minimising the convergence factor for all mesh ratios. Numerical results confirm the proximity of theoretical optimal β_{opt} to the optimal numerical parameter.

AMS subject classifications: 65M10, 78A48

Key words: Parareal method, time-varying problem, convergence analysis, parameter optimisation.

1. Introduction

A parareal method is an iterative algorithm characterised by two propagators $\mathscr{G}$ and $\mathscr{F}$, respectively, associated with large ΔT and small Δt step sizes, such that $\Delta T = J\Delta t$ and $J \geq 2$ is an integer. The $\mathscr{G}$-propagator is defined by a cheap, stable, low order time-integrator, such as backward Euler method, while the $\mathscr{F}$-propagator is defined by an expensive high order time-integrator. The algorithm proposed by Lions *et al.* [21] is widely used in Hamiltonian systems [1, 3, 8], parabolic equations [6, 24], first and second order hyperbolic equations [4, 7], PDE-constrained control and optimisation [5, 22, 23], singularly perturbed ODEs and PDEs [12, 18], Volterra integral equations [20, 38], time-periodic problems [9, 32], simulations of plasma turbulence [28, 29] and fractional PDEs [34, 37].

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The algorithm converges rapidly and robustly with respect to the change of mesh ratio J and discretisation parameters Δt and Δx. Therefore, it is possible to describe the solution at later stages without accurate information about earlier times, while the global accuracy of the method using only a few iterations is comparable to higher order expensive $\mathscr{F}$-propagators with fine time step sizes. In this work we use the backward Euler method as a $\mathscr{G}$-propagator in parareal algorithms for the following problem:

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = \alpha(t)(-\Delta)^{\eta} u(\mathbf{x},t) + f(\mathbf{x},t), \quad (\mathbf{x},t) \in \Omega \times (0,T), \tag{1.1}$$

where $\alpha(t) > 0, \eta \in (0,1)$ and $\Omega \subset \mathbb{R}^d$ with $d = 1,2,3$. Moreover, the spatial fractional Laplacian operator $(-\Delta)^{\alpha}$ is defined via an eigenfunction expansion on a finite-size spatial domain similar to [2,14,15,26,31,39,41]. It is worth noting that the definition of this operator based on Fourier transform [17,30] is less convenient from numerical point of view, although in both cases the matrix transform method provides an efficient spatial discretisation [14,25,39,40]. This discretisation of the fractional Laplacian $(-\Delta)^{\alpha}$ consists in establishing the matrix representation $\mathbf{A}$ for the approximation of the negative Laplacian $-\Delta$, which is then raised to the same fractional power α, thus obtaining $(-\Delta)^{\alpha} \approx \mathbf{A}^{\alpha}$. In this way, the matrix-vector product $A^{\alpha} b$ can be approximated by various numerical methods, including contour integrals [2], Lanczos method [31,39] and others — cf. Refs. [16,19,27].

Consider a mesh with m nodes and let $u_j(t)$ denote the value of a function $u(\mathbf{x},t)$ at j-th node $\mathbf{x}_j$. To find the approximate solution $\mathbf{U}(t)$ of (1.1), we apply the matrix transform method to the Eq. (1.1) and arrive at the system of ODEs

$$\mathbf{U}'(t) + \alpha(t)\mathbf{A}^{\eta}\mathbf{U}(t) = \mathbf{f}(t), \tag{1.2}$$

with a diagonalisable matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ such that $\sigma(\mathbf{A}) \subset [0,+\infty)$. The convergence of the parareal algorithm for the problem (1.1) with constant coefficient α is well studied. In particular, if $\mathscr{G}$ is the backward Euler method and $\mathscr{F}$ the exact-numerical-propagator — i.e. if $\mathscr{F} = e^{-\alpha\mathbf{A}^{\eta}\Delta t}$, then according to Fig. 1, the convergence factor $\rho := \max_{z\in\sigma(\Delta T \alpha \mathbf{A}^{\eta})} \mathscr{K}(z,J)$ satisfies the relation

$$\rho \approx \frac{1}{3} \quad \text{for all} \quad J \geq 2, \tag{1.3}$$

where $\sigma(\Delta T \alpha \mathbf{A}^{\eta})$ is the spectrum of $\Delta T \alpha \mathbf{A}^{\eta}$ and the term $\mathscr{K}(z,J)$, called the contraction factor of the parareal algorithm, is defined by

$$\mathscr{K}(z,J) = \frac{\left|\left(e^{-z/J}\right)^J - 1/(1+z)\right|}{1 - |1/(1+z)|} = \frac{\left|e^{-z} - 1/(1+z)\right|}{1 - |1/(1+z)|}.$$

The term $1/(1+z)$ is the stability function for backward Euler's method. We note however that in practical computations, the function $e^{-\alpha\mathbf{A}^{\eta}\Delta t}$ is not the best choice for $\mathscr{F}$-propagator, since it requires a special treatment [13]. If one uses a Runge-Kutta method as an $\mathscr{F}$-propagator, the contraction factor $\mathscr{K}$ has the form

$$\mathscr{K}(z,J) = \frac{\left|\mathscr{R}_f^J(z/J) - \mathscr{R}_g(z)\right|}{1 - |\mathscr{R}_g(z)|},$$