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Generalised Müntz Spectral Galerkin Methods for Singularly Perturbed Fractional Differential Equations

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Abstract. A family of orthogonal generalised Müntz-Jacobi functions is introduced and used for solving singularly perturbed fractional differential equations. Such basis functions can provide much better approximation for boundary layers or endpoint singularities than usual polynomial bases. The fractional integrals and derivatives of generalised Müntz-Jacobi functions are accurately calculated. The corresponding Petrov-Galerkin and Galerkin methods are very efficient. Numerical examples demonstrate a significant improvement in the accuracy of the methods.

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1. Introduction

It is well-known that compared to low-order methods, spectral methods based on global orthogonal polynomials provide a very accurate approximation of smooth solutions with significantly smaller degree of freedom — cf. [4, 9, 21] and references therein. However, they can lose their efficiency if solution exhibits sharp interfaces, strong corner singularities, thin internal and/or boundary layers. In such situations, the usual spectral methods based on Gauss-type grids often fail to produce approximations satisfying the requirements of real physics. A possible solution to this problem would be the use of adaptive method with mesh/grid refinement but the global character of spectral methods does not gracefully handle local mesh refinements. Even for grid adaptation with a moving mesh, the adaptive

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spectral approximations generated by moving mesh PDEs have to satisfy strong regularity conditions. Therefore, one usually employs a suitable preassigned mapping redistributing the grid points to where they are mostly needed [1,2,12,20]. For example, spectral methods with singular mapping techniques have been successfully used for resolving boundary layers — cf. Refs. [13, 14, 24, 26].

The approach here continues this line. We first show that the application of the singular mapping introduced in [26] to Jacobi polynomials with special parameters, produces mutually orthogonal non-polynomial functions, closely related to the Müntz-Jacobi functions [22] and fractional Jacobi polynomials [10]. The mapped Jacobi polynomials can be now considered as generalised Müntz-Jacobi functions. The present work also focuses on singularly perturbed fractional differential equations, the solutions of which may have sharp boundary layers and corner singularities. One of the main problems here is that the change of variables can lead to more complicated operators and additional singularities. In order to overcome this difficulty, we develop a very accurate scheme to evaluate the results of the action of fractional operators on generalised Müntz-Jacobi functions. Moreover, we consider efficient Petrov-Galerkin methods using special basis functions which can lead to diagonal or symmetric stiffness matrices and to better conditioned linear systems.

The rest of the paper is organised as follows. In Section 2, we introduce mapped Jacobi polynomials, consider their properties and connections with the Müntz-type functions and present an accurate scheme for the evaluation of their fractional integrals and derivatives. The next two sections are devoted to efficient Galerkin methods for singularly perturbed fractional initial value and boundary value problems. In addition, we also consider various examples aimed to show the advantage of the mapping technique proposed.

2. Mapped Jacobi Polynomials and Their Properties

In this section, we introduce mapped Jacobi polynomials and establish accurate formulas for computing the fractional integrals and derivatives of such aggregates. Moreover, we show the close connection of the mapped Jacobi polynomials to the Müntz-type functions. It turns out that the mapped Jacobi polynomials we study, can be considered as generalised Müntz-Jacobi functions. The approximation properties of the mapped Jacobi polynomials are also studied. In particular, we note an improved convergence order for functions with boundary layers and endpoint singularities.

2.1. Singular mappings

Let $\mathbb{N}$ and $\mathbb{R}$ be, respectively, the sets of positive integers and real numbers and let

$$\mathbb{N}_0 := \{0\} \cup \mathbb{N}, \quad \mathbb{R}^+ := \{a \in \mathbb{R} : a > 0\}, \quad \mathbb{R}_0^+ := \{0\} \cup \mathbb{R}^+. \tag{2.1}$$

Following [26], we consider the one-to-one mapping $g : [-1,1] \to [-1,1]$ defined by

$$x = g(y; r, l) = -1 + \sigma_{r,l} \int_{-1}^{y} (1-t)^r (1+t)^l \, dt, \quad r, l \in \mathbb{N}_0, \tag{2.2}$$