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An A-ϕ Scheme for Type-II Superconductors

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Abstract. A fully discrete $\boldsymbol{A}$-ϕ finite element scheme for a nonlinear model of type-II superconductors is proposed and analyzed. The nonlinearity is due to a field dependent conductivity with the regularized power-law form. The challenge of this model is the error estimate for the nonlinear term under the time derivative. Applying the backward Euler method in time discretisation, the well-posedness of the approximation problem is given based on the theory of monotone operators. The fully discrete system is derived by standard finite element method. The error estimate is suboptimal in time and space.

AMS subject classifications: 65M10, 78A48

Key words: Nonlinearity, finite element methods, well-posedness, convergence, error estimates.

1. Introduction

Superconductors can conduct the electric current freely without any resistances under the critical temperature. This magic character attracts many physicians and mathematicians. The superconductors can be mainly divided into two classes, the type-I and the type-II, depended on another important parameter, the critical magnetic field. The type-I superconductors are also called the Pippard conductors or the soft conductors, which is consist of the pure metals with one low (soft) critical field. The type-II ones are consist of the alloys with two critical fields, the lower one $\boldsymbol{H}_{c1}$ and the higher one $\boldsymbol{H}_{c2}$. The type-II superconductors get into the mixed state when the outside magnetic field holds that $\|\boldsymbol{H}_{c1}\| < \|\boldsymbol{H}\| < \|\boldsymbol{H}_{c2}\|$. During this period, some magnetic fluxes begin to penetrate into the type-II superconductors, and perform like a nonlinear diffusion process. Most of the high temperature superconductive materials belong to the type-II.

With being widely used in industry, it is very important to study the numerical methods to the macroscopic models of the type-II superconductors. The mathematical model for the type-II superconductors is derived from the Maxwell's equations by dropping the displacement current term, and replaced the linear Ohm's constitutive law with some certain nonlinear constitutive laws, such as the Bean's critical-state model [2], the Rhyner's power law [12] and so on (more overview of models refers to [6]).

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Let $\Omega \subset \mathbb{R}^3$ be a convex and smooth bounded domain with the connected boundary denoted by Γ. The eddy current model for the type-II superconductor states as follows [13],

$$\begin{cases} \nabla \times \boldsymbol{H} = \boldsymbol{J}_s + \boldsymbol{J} & \text{in } \Omega \times (0,T), \\ \nabla \times \boldsymbol{E} = -\partial_t(\mu \boldsymbol{H}) & \text{in } \Omega \times (0,T), \end{cases} \tag{1.1}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{J}_s$ is the source current density, $\boldsymbol{J}$ is the induced current density, and μ denotes the magnetic permeability of the medium. Suppose Ω is occupied by superconductive material Ω_c with boundary Γ_c, surrounded by insulating region Ω_{nc}. The conservation law of current density is as follows,

$$\boldsymbol{J}(\boldsymbol{E}) = \begin{cases} |\boldsymbol{E}|^{-\alpha} \boldsymbol{E}, & \text{in } \Omega_c, \\ 0, & \text{in } \Omega_{nc}, \end{cases} \tag{1.2}$$

with $\alpha \in (0,1)$. This power law describes the conductive relation of the type-II superconductors [14,15]. Further, we assume that μ is a piecewise positive constant in Ω, and there exist constants μ_{min} and μ_{max} such that $0 < \mu_{min} \le \mu \le \mu_{max}$.

In this paper, we consider the Maxwell's equations (1.1) with the power law nonlinear eddy current (1.2), and apply the $\boldsymbol{A}$-ϕ method to decompose the electric field into summation of a vector potential and a gradient of any scalar potential in the conductor, while we only need to solve the vector potential in the insulator. Although the $\boldsymbol{A}$-ϕ method may bring an extra unknown in the conductor, it has some nice properties:

- Both $\boldsymbol{A}$ and ϕ potentials are continuous, and the discontinuity of the electric field at the interface of different media is characterized by the gradient of the scalar potential ϕ [7].
- The $\boldsymbol{A}$-ϕ method can be applied to both simply and multiply connected conductor [4]. Moreover, the convexity or smoothness of the conductors Ω_c is no more needed.
- Under the same mesh partition, the number of edges is usually more than the number of nodes. The computational cost of the nodal $\boldsymbol{A}$-ϕ method is no more than the one of the edge element method [9].
- The $\boldsymbol{A}$-ϕ nodal element method takes advantages from other disciplines and many popular finite element packages and computational techniques can be applied directly, such as natural coupling to moment and boundary element methods, global energy conservation [3,11].

The $\boldsymbol{A}$-ϕ method has been widely used in electrical engineering and its superiors have been demonstrated by practical applications [4,7,8,10,13]. However, there are rare literatures for rigorous mathematical theories on the $\boldsymbol{A}$-ϕ method, especially for nonlinear Maxwell's equations. This is the motivation of our work.

By Decomposing the electric field $\boldsymbol{E}$ into $\boldsymbol{E} = \boldsymbol{A} + \nabla \phi$, where $\boldsymbol{A}$ is a vector potential, and