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A Fourth-Order Compact Finite Difference Scheme for Higher-Order PDE-Based Image Registration

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Abstract. Image registration is an ill-posed problem that has been studied widely in recent years. The so-called curvature-based image registration method is one of the most effective and well-known approaches, as it produces smooth solutions and allows an automatic rigid alignment. An important outstanding issue is the accurate and efficient numerical solution of the Euler-Lagrange system of two coupled nonlinear biharmonic equations, addressed in this article. We propose a fourth-order compact (FOC) finite difference scheme using a splitting operator on a 9-point stencil, and discuss how the resulting nonlinear discrete system can be solved efficiently by a nonlinear multi-grid (NMG) method. Thus after measuring the h-ellipticity of the nonlinear discrete operator involved by a local Fourier analysis (LFA), we show that our FOC finite difference method is amenable to multi-grid (MG) methods and an appropriate point-wise smoothing procedure. A high potential point-wise smoother using an outer-inner iteration method is shown to be effective by the LFA and numerical experiments. Real medical images are used to compare the accuracy and efficiency of our approach and the standard second-order central (SSOC) finite difference scheme in the same NMG framework. As expected for a higher-order finite difference scheme, the images generated by our FOC finite difference scheme prove significantly more accurate than those computed using the SSOC finite difference scheme. Our numerical results are consistent with the LFA analysis, and also demonstrate that the NMG method converges within a few steps.

AMS subject classifications: 65F10, 65M55, 68U10

Key words: Curvature image registration, fourth-order compact finite difference scheme, local Fourier analysis, nonlinear multi-grid method, nonlinear biharmonic equation.

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1. Introduction

Image registration is a fundamental problem in the field of image analysis. The problem arises when two or more images are taken at different times, or from different viewpoints or by different sensors, and then must be compared and integrated into one representation so that the information can be accessed readily and accurately. Over the years, image analysis has been used routinely in medical diagnosis, treatment guidance and monitoring. Medical applications are surveyed in Refs. [31,43,44] and references therein.

Higher-order partial differential equation (PDE) based image registration methods have proven to be very valuable in a wide range of applications [16–18,24,25,33–35,37,38], but the numerical solutions often need to be improved. In deriving the associated PDE in such methods, we may let $\Omega \subset \mathbb{R}^d$ denote an image domain and $R, T : \Omega \to \mathbb{R}$ two images of the same scene, respectively referred to as the *reference* and *template* image. The problem is to compute an optimal *deformation* or *displacement field*

$$\boldsymbol{u} : \mathbb{R}^d \to \mathbb{R}^d, \qquad \boldsymbol{u} : \mathbf{x} \mapsto \boldsymbol{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}), \cdots, u_d(\mathbf{x}))^\top$$

such that the transformed template $T(\mathbf{x} + \boldsymbol{u}(\mathbf{x})) = T(\boldsymbol{u})$ becomes similar to the reference R. It is also assumed that the given images are smooth, and compactly supported functions map the image domain Ω into $V \subset [0, \infty)$, where $d \in \mathbb{N}$ represents the spatial dimension of the images — usually $d = 2$ (images) or $d = 3$ (volume data set) with boundary $\partial\Omega$. Here we consider the two-dimensional case ($d = 2$), and that $\Omega = [0,1]^2 \subset \mathbb{R}^2$ and $V = [0,1]$ for grey intensity images.

If the image intensities of R and T are comparable, an obvious method for computing the unknown deformation $\boldsymbol{u}$ would be minimisation of the sum of squared differences:

$$\min_{\boldsymbol{u}} \left\{ \mathscr{D}(\boldsymbol{u}) = \frac{1}{2} \int_\Omega (T(\mathbf{x} + \boldsymbol{u}(\mathbf{x})) - R(\mathbf{x}))^2 \, d\mathbf{x} \right\}. \tag{1.1}$$

Since the minimisation of $\mathscr{D}$ is generally an ill-posed problem in the sense of Hadamard, a regularisation technique is used to impose a constraint on the solution $\boldsymbol{u}$ by a *regulariser* $\mathscr{R}$ that penalises unreasonable and irregular solutions from some *a priori* knowledge. Thus the image registration problem can be posed as the minimisation problem of a joint energy functional

$$\min_{\boldsymbol{u} \in \mathscr{U}} \{ \mathscr{J}_\alpha[\boldsymbol{u}] = \mathscr{D}(\boldsymbol{u}) + \alpha \mathscr{R}(\boldsymbol{u}) \} \tag{1.2}$$

involving the regularisation parameter $\alpha > 0$ that compromises similarity and regularity, with $\boldsymbol{u}$ searched over a set $\mathscr{U}$ of admissible functions minimising $\mathscr{J}_\alpha$. For two reasons, we select the curvature-based regulariser

$$\mathscr{R}(\boldsymbol{u}) = \frac{1}{2} \sum_{l=1}^{2} \int_\Omega (\Delta u_l)^2 d\mathbf{x} \tag{1.3}$$

introduced by Fischer & Modersitzki [24]. Firstly, Δu_l in the integrand may be viewed as an approximation to the mean curvature of the surface of u_l in $\mathbb{R}^3$ since

$$\kappa_M(u_l) = \nabla \cdot \frac{\nabla u_l}{\sqrt{1 + |\nabla u_l|^2}} = \frac{(1 + u_{l_{x_1}}^2) u_{l_{x_1 x_1}} - 2 u_{l_{x_1}} u_{l_{x_2}} u_{l_{x_1 x_2}} + (1 + u_{l_{x_2}}^2) u_{l_{x_2 x_2}}}{(1 + u_{l_{x_1}}^2 + u_{l_{x_2}}^2)^{3/2}}.$$