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An Inverse Diffraction Problem: Shape Reconstruction

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Abstract. An inverse diffraction problem is considered. Both classical Tikhonov regularisation and a slow-evolution-from-the-continuation-boundary (SECB) method are used to solve the ill-posed problem. Regularisation error estimates for the two methods are compared, and the SECB method is seen to be an improvement on the classical Tikhonov method. Two numerical examples demonstrate their feasibility and efficiency.

AMS subject classifications: 65J20, 65R35
Key words: Inverse diffraction problem, ill-posed problems, Tikhonov regularisation, stability estimate, error estimate, SECB.

1. Introduction

The reconstruction problem for an aperture in the plane from the resulting diffraction pattern arises in acoustics and optics, and in this article we consider the following inverse diffraction problem.

Let D be a bounded aperture in an infinite perfectly soft screen located in the plane $z = 0$ in $\mathbb{R}^3$. An harmonic plane wave with wavenumber k propagates along with the positive z direction, and on hitting the screen it escapes through the aperture D. The measured data at the receiving screen at $z = d > 0$ is given. The problem is to reconstruct the shape (domain) of the aperture D— cf. Fig. 0. This problem has been considered by Sondhi [1], Bertero [2] and Magnanini [3]. Under the Kirchhoff approximation, the mathematical modelling of this phenomenon can be formulated as follows:

Problem 1. *If $u(x,y,z)$ denotes the solution of the boundary value problem*

$$u_{xx} + u_{yy} + u_{zz} + k^2 u(x,y,z) = 0\,, \qquad (x,y) \in \mathbb{R}^2, z > 0\,, \tag{1.1}$$

$$u(x,y,d) = g(x,y)\,, \qquad (x,y) \in \mathbb{R}^2\,, \tag{1.2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0\,,\ \textit{where}\ r = \sqrt{x^2+y^2+z^2}\,, \tag{1.3}$$

we seek to determine $u(x,y,0) = \chi_D(x,y)$ from the data $g(x,y)$ where $\chi_D(x,y)$ is the characteristic function of the domain D.

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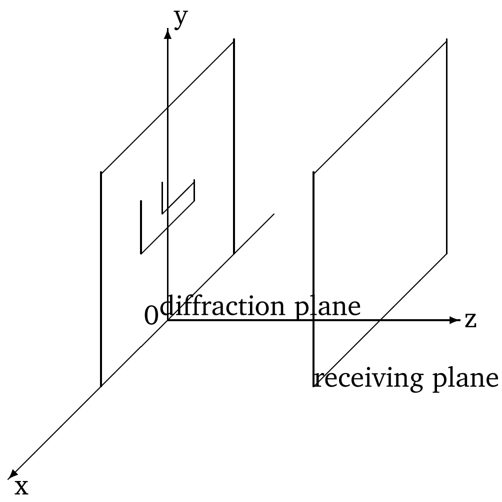

Figure 1: Inverse diffraction problem.

However, here we will consider that the characteristic function χ_D is replaced by an $\mathbb{R}^2$-square integrable function $f(x,y)$, in the following generalisation of Problem 1:

Problem 2. *If $u(x,y,z)$ denotes the solution to the boundary value problem*

$$u_{xx}+u_{yy}+u_{zz}+k^2u(x,y,z)=0\,, \qquad (x,y)\in\mathbb{R}^2, z>0\,, \tag{1.4}$$

$$u(x,y,d)=g(x,y)\,, \qquad (x,y)\in\mathbb{R}^2\,, \tag{1.5}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r}-iku\right)=0\,,\ \textit{where}\ r=\sqrt{x^2+y^2+z^2}\,, \tag{1.6}$$

we seek to determine $u(x,y,0)=f(x,y)$ from the data $g(x,y)$.

Before investigating the inverse problem, let us briefly discuss the corresponding forward problem of Problem 2 as follows.

Forward problem of Problem 2. *Find the solution $u(x,y,z)$ of the boundary value problem*

$$u_{xx}+u_{yy}+u_{zz}+k^2u(x,y,z)=0\,, \qquad (x,y)\in\mathbb{R}^2, z>0\,, \tag{1.7}$$

$$u(x,y,0)=f(x,y), \qquad (x,y)\in\mathbb{R}^2\,, \tag{1.8}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r}-iku\right)=0\,,\textit{where}\ r=\sqrt{x^2+y^2+z^2}\,. \tag{1.9}$$

We wish to find the $u(x,y,z)=g(x,y)$ from the data $f(x,y)$ for a fixed $0<z\le d$. It is well known that if $f(x,y)$ has bounded support, then the forward problem admits a unique solution

$$u(x,y,z)=\int_{\mathbb{R}^2} H_z(x-\mu,y-\nu)f(\mu,\nu)d\mu d\nu := A(z)f(x,y)\,, \tag{1.10}$$