East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.310315.070815a

Vol. **5**, No. 4, pp. 301-311
November 2015

An L^∞-error Estimate for the h-p Version Continuous Petrov-Galerkin Method for Nonlinear Initial Value Problems

Lijun Yi*

Department of Mathematics, Shanghai Normal University, Shanghai 200234; and Division of Computational Science, E-institute of Shanghai Universities, Shanghai 200234, China.

Received 31 March 2015; Accepted (in revised version) 7 August 2015.

Abstract. The h-p version of the continuous Petrov-Galerkin time stepping method is analyzed for nonlinear initial value problems. An L^∞-error bound explicit with respect to the local discretization and regularity parameters is derived. Numerical examples are provided to illustrate the theoretical results.

AMS subject classifications: 65L60, 65L05, 65L70

Key words: Initial value problems, h-p version, time stepping method, continuous Petrov-Galerkin method, error bound.

---

1. Introduction

This article discusses an h-p version of the continuous Petrov-Galerkin (CPG) method, introduced by Wihler [13] to solve nonlinear initial value problems for ordinary differential equations of the form

$$\begin{cases} u'(t) = f(t, u(t)) , & t \in I := (0, T) , \\ u(0) = u_0 \in \mathbb{R} , \end{cases} \tag{1.1}$$

where $u : \bar{I} \to \mathbb{R}$ and $f = f(t, u) : \bar{I} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. We assume that f is uniformly Lipschitz continuous with respect to u — i.e.

$$|f(t, u) - f(t, v)| \le L|u - v| \qquad \forall\, u, v \in \mathbb{R} ,\ t \in \bar{I} , \tag{1.2}$$

for some Lipschitz constant $L > 0$. Under the assumption (1.2), there exists a unique solution u of (1.1) that is continuously differentiable.

The origins of Galerkin methods for the numerical solution of ordinary differential equation (ODE) initial value problems can be traced back to the 1970s — e.g. continuous

---

*Corresponding author. *Email address:* `ylj5152@shnu.edu.cn` (L. Yi)

Galerkin (CG) approximations [4,6,7], where global adaptive error control was addressed, and discontinuous Galerkin (DG) methods — cf. [2,3] and references therein. Both the CG and DG time stepping methods are implicit single-step schemes that allow for arbitrary variation in the time step and the approximation order, which naturally lead to an h-p version Galerkin framework. Recently, the h-p version CG and DG methods were analysed for nonlinear initial value ODE [10,13], where exponential rates of convergence were achieved for solutions with start-up singularities. Moreover, the h-p version CG and DG methods have been applied successfully to Volterra integro-differential equations (VIDE) [1, 14], linear parabolic problems [11], and VIDE of parabolic type [9].

The aim of this article is to derive an L^∞-error estimate for the h-p version of the CPG method for the problem (1.1). An error analysis in the L^2- and H^1-norm has previously been carried out for the h-p version CPG method for nonlinear initial value problems [13] where an error splitting $u-U=(u-\widetilde{U})+(\widetilde{U}-U)$ was employed, with U denoting the h-p approximation to the exact solution u and $\widetilde{U}$ the solution of an auxiliary variational problem — cf. (3.12) below. However, it seems that the derivation of an L^∞-error bound for the term $u-\widetilde{U}$ is not a trivial task, and here the error splitting is approached from a different perspective. Thus the function $\widetilde{U}$ can be interpreted in another way, coinciding with a well-studied polynomial approximation $\mathscr{I}u$ — cf. (3.4) and Lemma 3.2 below. The main task in the error analysis can then be reduced to estimating only the term $\widetilde{U}-U$. This leads to an L^∞-error bound explicit with respect to the time steps, the approximation orders, and the regularity of the exact solution. In Section 2, the h-p version CPG time stepping method for (1.1) is introduced, and the error analysis and an L^∞-error estimate for the CPG method are obtained in Section 3. Numerical examples illustrating the performance of the CPG method are presented in Section 4, and brief concluding remarks are made in Section 5.

2. Formulation of the h-p Version CPG Method

To describe the CPG method for the discretisation of the problem (1.1), consider a partition $\mathscr{T}_h$ of the time interval I given by the points

$$0=t_0<t_1<t_2<\cdots<t_{N-1}<t_N=T\ .$$

Let $I_n=(t_{n-1},t_n)$ and $k_n=t_n-t_{n-1}$ for $1\le n\le N$, and denote by $k=\max\{k_n\}_{n=1}^N$ the maximum step-size of the partition $\mathscr{T}_h$. In particular, if there is a constant $C_q>0$ such that $k\le C_q k_n$ for all $1\le n\le N$, the partition $\mathscr{T}_h$ is called quasi-uniform. An approximation order $r_n\ge 1$ is assigned to each time interval I_n, and stored in the polynomial degree vector $\mathbf{r}=\{r_n\}_{n=1}^N$. The tuple $(\mathscr{T}_h,\mathbf{r})$ is called an h-p discretisation of I.

For a given h-p discretisation $(\mathscr{T}_h,\mathbf{r})$ of I, the h-p approximation and test function spaces are respectively

$$S^{\mathbf{r},1}(\mathscr{T}_h)=\left\{u\in H^1(I):u|_{I_n}\in P_{r_n}(I_n),1\le n\le N\right\}$$

and

$$S^{\mathbf{r}-1,0}(\mathscr{T}_h)=\left\{u\in L^2(I):u|_{I_n}\in P_{r_n-1}(I_n),1\le n\le N\right\}\ ,$$