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Features of the Nyström Method for the Sherman-Lauricella Equation on Piecewise Smooth Contours

Victor D. Didenko[1,*] and Johan Helsing[2]

[1] *Faculty of Science, University of Brunei Darussalam, Bandar Seri Begawan, BE1410 Brunei.*
[2] *Numerical Analysis, Centre for Mathematical Sciences, Lund University, Box 118, SE-221 00 Lund, Sweden.*

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Abstract. The stability of the Nyström method for the Sherman-Lauricella equation on contours with corner points c_j, $j=0,1,\cdots,m$ relies on the invertibility of certain operators A_{c_j} belonging to an algebra of Toeplitz operators. The operators A_{c_j} do not depend on the shape of the contour, but on the opening angle θ_j of the corresponding corner c_j and on parameters of the approximation method mentioned. They have a complicated structure and there is no analytic tool to verify their invertibility. To study this problem, the original Nyström method is applied to the Sherman-Lauricella equation on a special model contour that has only one corner point with varying opening angle θ_j. In the interval $(0.1\pi, 1.9\pi)$, it is found that there are 8 values of θ_j where the invertibility of the operator A_{c_j} may fail, so the corresponding original Nyström method on any contour with corner points of such magnitude cannot be stable and requires modification.

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Key words: Sherman–Lauricella equation, Nyström method, stability.

1. Introduction

Let Γ be a simple closed positively oriented contour in the complex plane $\mathbb{C}$. The Sherman–Lauricella equation

$$\omega(t)+\frac{1}{2\pi i}\int_\Gamma \omega(\tau)\,d\ln\left(\frac{\tau-t}{\overline{\tau}-\overline{t}}\right)-\frac{1}{2\pi i}\int_\Gamma \overline{\omega(\tau)}\,d\left(\frac{\tau-t}{\overline{\tau}-\overline{t}}\right)=f(t),\quad t\in\Gamma, \tag{1.1}$$

where the bar denotes the complex conjugation and ω is an unknown function, plays an important role in various fields of applied mathematics — including elasticity theory, theory of incompressible flows, radar imaging [11–14]. However, at present there is no

*Corresponding author. *Email addresses:* `diviol@gmail.com` (V. D. Didenko), `helsing@maths.lth.se` (J. Helsing)

general analytic solution of Eq. (1.1) available. If the contour Γ is smooth, then the integral operators in Eq. (1.1) are compact and the corresponding approximation methods for this equation can be studied without serious difficulties. On the other hand, when the contour Γ has corner points $c_1, c_2, \cdots, c_m$ the stability of the approximation method under consideration usually depends on the invertibility of certain operators $A_{c_0}, A_{c_1}, \cdots, A_{c_{m-1}}$ associated with the method itself and with the parameters of the corner points at hand. As a rule, such operators have a complicated structure, so their invertibility cannot be treated effectively. Nevertheless, apart from the approximation method each operator A_{c_j}, $j = 0, 1, \cdots, m-1$ does not depend on the shape of the contour Γ but on specific parameters of the corner point c_j, so the invertibility of such operators can be studied via connections with the stability of corresponding approximation methods considered on certain special model curves.

In the present paper, we investigate this property for the Nyström method of Ref. [3], and find that in the interval $(0.1\pi, 1.9\pi)$ there are angles for which the operators A_{c_j} are not invertible. The Nyström method and operators A_{c_j} are discussed in Section 2, and the numerical simulations presented in Section 3 are followed by our final summary and discussion in Section 4.

2. The Nyström Method and the Operators A_{c_j}

Let $\gamma = \gamma(s)$ be a 1-periodic parametrization of Γ. For the sake of simplicity, let us assume that $c_j = \gamma(j/m)$ for all $j = 0, 1, \cdots, m-1$, the function γ is two times continuously differentiable on each interval $(j/m, (j+1)/m)$ and

$$\left|\gamma'\left(\frac{j}{m}+0\right)\right| = \left|\gamma'\left(\frac{j}{m}-0\right)\right|, \quad j = 0, 1, \cdots, m-1.$$

Let us now construct a mesh that will be used in the following discussion. Set $n = qm$ for $q = 1, 2, \cdots$, and for such n note that any corner of Γ is always an end point of a subinterval $(\gamma(r/n), \gamma((r+1)/n))$. Let d be a positive integer and let $0 < \varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_{d-1} < 1$ and $0 < \delta_0 < \delta_1 < \cdots < \delta_{d-1} < 1$ be real numbers. Consider two sets of points on Γ — viz.

$$\tau_{lp} = \gamma\left(\frac{l+\varepsilon_p}{n}\right), \quad t_{lp} = \gamma\left(\frac{l+\delta_p}{n}\right), \quad l = 0, 1, \cdots, n-1; p = 0, 1, \cdots, d-1. \tag{2.1}$$

According to [3], the approximate values $\omega(\tau_{lp})$ of an exact solution ω of Eq. (1.1) at the points τ_{lp} are defined by the following system of algebraic equations:

$$\begin{aligned}
\omega(\tau_{kr}) &+ \frac{1}{2\pi i}\sum_{l=0}^{n-1}\sum_{p=0}^{d-1} w_p \omega(\tau_{lp}) \left(\frac{\tau'_{lp}}{\tau_{lp}-t_{kr}} - \frac{\overline{\tau'}_{lp}}{\overline{\tau}_{lp}-\overline{t}_{kr}}\right)\frac{1}{n} \\
&- \frac{1}{2\pi i}\sum_{l=0}^{n-1}\sum_{p=0}^{d-1} w_p \overline{\omega(\tau_{lp})} \left(\frac{1}{\overline{\tau}_{lp}-\overline{t}_{kr}}\frac{\tau'_{lp}}{n} - \frac{\tau_{lp}-t_{kr}}{(\overline{\tau}_{lp}-\overline{t}_{kr})^2}\frac{\overline{\tau'}_{lp}}{n}\right)
\end{aligned}$$