East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.101117.130418

Vol. **8**, No. 3, pp. 586-597
August 2018

# An $\ell_q$- Seminorm Variational Model for Impulse Noise Reduction

Yoon Mo Jung[1], Taeuk Jeong[2] and Sangwoon Yun[3,*]

[1] *Department of Mathematics, Sungkyunkwan University, Jangan-gu, Suwon, Gyeonggi-do 16419, Republic of Korea.*
[2] *Department of Computational Science and Engineering, Yonsei University 50 Yonsei-Ro, Seodaemun-Gu, Seoul 03722, Republic of Korea.*
[3] *Department of Mathematics Education, Sungkyunkwan University, Seoul 03063, Republic of Korea.*

*Received 10 November 2017; Accepted (in revised version) 13 April 2018.*

**Abstract.** A variational $\ell_q$-seminorm model to reduce the impulse noise is proposed. For $0 < q < 1$, it captures sparsity better than the $\ell_1$-norm model. Numerical experiments show that for small $q$ this model is more efficient than TV$\ell_1$ model if the noise level is low. If the noise level grows, the best possible parameter $q$ in the model approaches 1.

**AMS subject classifications**: 94A08, 90C26

**Key words**: Impulse noise, sparsity, $\ell_q$-seminorm, total variation, iterative reweighted algorithm.

## 1. Introduction

Image denoising aims to restore an original image from its noise-corrupted image. It is one of the most fundamental problem in image processing. Inspecting governing physics and various circumstances, noise is usually modelled by using probability distributions such as Gaussian and Poisson distributions. In this paper, we consider impulsive noise, which is modelled as an unipole or bipole shape distribution [9]. Impulsive noise arises from malfunctioning camera sensors, faulty memory locations, or noisy channel transmissions [1].

The median filters are popular spatial filters due to their low computational cost and the ability to treat impulsive noise as an outlier [9]. In order to deal with high level noise and preserve the details and edges of the original image, the median-type filters use adaptivity, multistate or homogeneity information often combined with regularisation. For more information we refer the reader to Chan *et al.* [1] and references therein.

Another important class of impulsive noise reduction approaches comprises variational methods. It is well known that $\ell_2$-norm based data-fidelity terms do not succeed in removing impulse noise. They have a heavy tailed distribution and therefore the $\ell_1$-norm is used

*Corresponding author. *Email addresses:* yoonmojung@skku.edu (Y.M. Jung), iamlogin@yonsei.ac.kr (T. Jeong), yswmathedu@skku.edu (S. Yun)

 

instead [1,11,18]. Let us remark that the median filter is a minimiser of the mean absolute error, which is closely related to the $\ell_1$-minimisation. Analogously, as image prior one can choose an edge-preserving regularisation functional [18].

The $\ell_1$-norm is widely used in compressed sensing because of its sparsity-inducing properties [2,5–7]. On the other hand, for $0 < q < 1$ the $\ell_q$-seminorm provides even more possibilities to exploit sparsity [8,15,16]. Thus we introduce the $\ell_q$-seminorm as a data-fidelity term and show numerically the relation between the sparsity level of impulse noise and a proper value of $q$, viz. if the sparsity level decreases, the best value of $q$ moves away from 1. The model we propose is non-differentiable and even non-convex and so it is challenging to solve it. To overcome these difficulties, we adapt an iteratively reweighted algorithm (IRA) — cf. Refs. [3,12].

This paper is organised as follows. A variational model with the $\ell_q$-seminorm aiming to reduce the impulse noise is introduced in Section 2. In Section 3, we describe an iteratively reweighted algorithm to solve the model and to study its convergence. Numerical experiments presented in Section 4, show the connection between noise level and the parameter $q$. Our conclusions are given in Section 5.

## 2. An $\ell_q$-Denoising Model for Impulsive Noise

Two common types of impulsive noise are salt-and-pepper noise and random valued impulse noise (RVIN). In the case of salt-and-pepper noise, corrupted pixels can take only the minimal or maximal intensity value, but the values between minimal and maximal ones are randomly chosen for RVIN. In both cases, the remaining pixels are unaffected [1,18]. If $f$ denotes a noisy observation of the original image $u$, it can be written as

$$f = u + n,$$

where $n$ is impulse noise. Discretised original and noisy images are represented by $M \times N$ matrices in $\mathbb{R}^{MN}$, so that $n$ is an $M \times N$ sparse matrix.

Typical variational denoising model consists of image prior $\mathscr{E}$ and data-fidelity $\mathscr{F}$ terms — viz.

$$\min_u c\mathscr{E}[u] + \mathscr{F}[u|f],$$

where $c > 0$ balances these terms. The $\ell_1$-norm is well-adapted to data-fidelity term because of the sparsity of the impulse noise [1,11,18]. If the original image is modeled by the function of bounded variation [14], then one can consider the following TV$\ell_1$ model [18]:

$$\min_{u \in U} c\,\mathrm{TV}(u) + \| u - f \|_1, \tag{2.1}$$

where $\mathrm{TV}(u) = \|\nabla u\| = \sum_{i,j} \sqrt{((u_x)_{i,j})^2 + ((u_y)_{i,j})^2}$, whereas $\nabla u$ and $u_x$, $u_y$ are discretised versions of gradient and partial derivatives of $u$, respectively. By $U$ we denote the admissible class $[0, C]^{MN}$ with $0 < C < \infty$, where $C$ is often set to 255 while considering the image values as 8 bit unsigned integers in the range $[0, 255]$ or 1 if scaled. The $\|\cdot\|_1$ in the second term is the $\ell_1$-norm of the space $\mathbb{R}^{MN}$ with the vectors $u$ and $f$ in $\mathbb{R}^{MN}$.