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Dynamics of Solitary Waves and Periodic Waves in a $(3+1)$-Dimensional Nonlinear Evolution Equation

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Abstract. The Hirota bilinear method is applied to a generalised $(3+1)$-dimensional nonlinear evolution equation. Using the Riemann theta function, we construct periodic wave solutions of the Eq. (1.1) and discuss their properties. Graphic examples show the propagation of the corresponding waves for different sets of parameters. We also study the asymptotic of periodic waves and their relation with solution solutions.

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1. Introduction

Nonlinear evolution equations (NLEEs) represent various phenomena in physics. Finding exact solutions of such equations is an important and tough problem. There are several approaches to NLEEs, including Hirota bilinear method [17], Darboux transformation [23], inverse scattering transformation [3], Lie group method [2] and so on. Further development of these methods allows to study the integrability of various classes of NLEEs and to derive their exact solutions – cf. Refs. [5, 16, 18, 21, 25, 26, 55, 58]. In 1980,

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Nakamura [31] proposed a method to construct quasi-periodic solutions of NLEEs. Lambert *et al.* [14, 19] used Bell polynomials [1], to investigate bilinear Backlund transformations (BTs) and Lax pairs of integrable equations. This approach, the Riemann theta function and the Hirota bilinear method produced various periodic wave solutions of NLEEs [4, 7–9, 20, 22, 27–29, 33, 36–40, 49].

On the other hand, higher-dimensional integrable models become more and more popular in mathematical physics, and our goal here is to study a (3+1)-dimensional generalised nonlinear evolution equation (GNEE)

$$\begin{aligned} &u_{yt}+\alpha u_{xz}+\beta u_{xxxy}+\gamma\left(uu_{xy}+u_xu_y+u_{xx}\omega+u_x\omega_x\right)=0,\\ &u_y=\omega_x, \end{aligned} \tag{1.1}$$

where $u=u(x,y,z,t)$ and α,β,γ are non-zero constants. This equation is connected with various real world problems and its exact solution can provide important information. If $\alpha=-3/2$, $\beta=1/2$, $\gamma=-1$, the Eq. (1.1) can be reduced to a (3+1)-dimensional nonlinear evolution equation

$$3u_{xz}-2u_{ty}-u_{xxxy}+2uu_{xy}+2u_xu_y+2\left(u_x\partial_x^{-1}u_y\right)_x=0.$$

Algebraic geometrical solutions of this equation have been studied in Refs. [15, 32]. On the other hand, to the best of the authors' knowledge, periodic wave solutions of the Eq. (1.1) have not been reported so far. In this paper, we construct soliton and periodic wave solutions of (1.1) and establish a connection between these two solution families. More precisely, in Section 2 we obtain a bilinear form and N-soliton solutions of the Eq. (1.1). Section 3 deals with new exact solutions found via maple symbolic computation software. The periodic wave solutions of the Eq. (1.1) are constructed in Section 4, and Section 5 describes relations between soliton and periodic wave solutions. Our conclusions are in Section 6.

2. A Bilinear Form and Soliton Solutions

Let us start with two important transformations

$$u=m(t)q_{xx},\quad \omega=m(t)q_{xy}, \tag{2.1}$$

where $m=m(t)$ is an unknown function to be determined later. Substituting (2.1) into (1.1) and integrating the resulting equation twice with respect to x, we obtain

$$E(q)=q_{yt}+\alpha q_{xz}+\beta\left(q_{xxxy}+3q_{xx}q_{xy}\right)=\theta, \tag{2.2}$$

under the constraint

$$m=3\beta/\gamma.$$

Using Refs. [1, 14, 19, 36–40], we rewrite the Eq. (2.2) as

$$E(q)=P_{yt}+\alpha P_{xz}+\beta P_{xxxy}=\theta, \tag{2.3}$$

where θ is an integrable constant. In the special case $\theta=0$, the Eq. (2.3) can be written via a bilinear form. More precisely, the following theorem holds.