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Optimal Error Estimates for a Fully Discrete Euler Scheme for Decoupled Forward Backward Stochastic Differential Equations

Bo Gong[1] and Weidong Zhao[2,*]

[1] *Department of Mathematics, Hong Kong Baptist University, Hong Kong, China.*
[2] *School of Mathematics & Finance Institute, Shandong University, Jinan, Shandong 250100, China.*

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Abstract. In error estimates of various numerical approaches for solving decoupled forward backward stochastic differential equations (FBSDEs), the rate of convergence for one variable is usually less than for the other. Under slightly strengthened smoothness assumptions, we show that the fully discrete Euler scheme admits a first-order rate of convergence for both variables.

AMS subject classifications: 60H35, 65C30

Key words: Forward backward stochastic differential equations, fully discrete scheme, error estimate

1. Introduction

On a filtered complete probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leqslant t\leqslant T}, P)$ where $\mathscr{F}_t$ is generated by the standard Brownian motion W_s, $0 \leqslant s \leqslant t$ and T is a fixed time horizon, we consider the decoupled forward-backward stochastic differential equations (FBSDEs):

$$dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dW_t, \quad X_0 = x_0\,, \tag{1.1}$$

$$-dy_t = f(t,X_t,y_t,z_t)\,dt - z_t\,dW_t, \quad y_T = g(X_T), \tag{1.2}$$

where $X_0 = x_0$ is the initial condition of the forward equation, $y_T = g(X_T)$ is the terminal condition of the backward equation, and b, σ, f and g are deterministic functions in $\mathbb{R}$. A triple $(X_t, y_t, z_t) : [0,T] \times \Omega \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is called a solution of Eqs. (1.1) and (1.2) if

*Corresponding author. *Email addresses:* `13479245@life.hkbu.edu.hk` (B. Gong), `wdzhao@sdu.edu.cn` (W. Zhao)

its components are $\mathscr{F}_t$-adapted, square integrable, and satisfy the respective forward and backward integral equations

$$X_t = X_0 + \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dW_s\,, \quad t \in [0,T],$$
$$y_t = g(X_T) + \int_t^T f(s,X_s,y_s,z_s)\,ds - \int_t^T z_s\,dW_s\,, \quad t \in [0,T],$$

where the two integrals with respect to the Brownian motion W_s are Itô-type.

Pardoux & Peng [10] proved the existence and uniqueness of the solution of nonlinear backward stochastic differential equations (BSDEs) with the $\mathscr{F}_T$-measurable terminal condition ξ, and Peng [12] gave the following probabilistic representation for the nonlinear Feynman-Kac solutions of Eqs. (1.1) and (1.2):

$$y_t = u(t,X_t), \qquad z_t = \partial_x u(t,X_t)\sigma(t,X_t), \quad t \in [0,T), \tag{1.3}$$

where $u(t,x)$ is the smooth solution of the partial differential equation

$$\partial_t u(t,x) + b(t,x)\partial_x u(t,x) + \frac{1}{2}\sigma(t,x)^2\partial_{xx}u(t,x) = -f\big(t,x,u(t,x),\partial_x u(t,x)\sigma(t,x)\big),$$

with the terminal condition $u(T,x) = g(x)$. Subsequently, FBDSEs have been studied extensively and applied in many fields — e.g. mathematical finance, stochastic optimal control, nonlinear expectation, risk measure, and related problems [4, 5, 11, 13]. It is very difficult to find solutions in explicit closed form, so considerable attention has been paid to the numerical solution of FBSDEs and many numerical schemes have already been proposed [1–3, 6, 14–21]. Here we reconsider the fully discrete Euler scheme for decoupled FBSDEs proposed in Ref. [14], and under certain regular conditions on the data b, σ, f and g we prove its first-order sup-norm convergence in solving Eqs. (1.1) and (1.2). In Section 2, some preliminaries are introduced, and the fully discrete Euler scheme is discussed in Section 3. Our error estimates of the scheme are derived in Section 4, and some concluding remarks are made in Section 5.

2. Preliminaries

Let us first list some notation — viz.

- $C_b^{\ell,k,k,k}$: the set of continuously differentiable functions $\psi : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with uniformly bounded partial derivatives $\partial_t^{\ell_1}\partial_x^{k_1}\partial_y^{k_2}\partial_z^{k_3}\psi$ for $0 \leqslant \ell_1 \leqslant \ell$ and $0 \leqslant k_1 + k_2 + k_3 \leqslant k$. Analogous definition applies for $C_b^{\ell,k}$.
- $(X_t^{r,x}, y_t^{r,x}, z_t^{r,x})$: solution (X_t, y_t, z_t) of (1.1), (1.2) with initial condition replaced by $X_r = x$, for $r \leqslant t$. And $(X_t^x, y_t^x, z_t^x) := (X_t^{t,x}, y_t^{t,x}, z_t^{t,x})$.
- $\mathscr{F}_t^{r,x}$: the σ-field generated by $\{X_s^{r,x} \mid r \leqslant s \leqslant t\}$, for $r \leqslant t$.