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Distribution of Discrete Time Delta-Hedging Error via a Recursive Relation

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Abstract. We introduce a new method to compute the approximate distribution of the Delta-hedging error for a path-dependent option, and calculate its value over various strike prices via a recursive relation and numerical integration. Including geometric Brownian motion and Merton's jump diffusion model, we obtain the approximate distribution of the Delta-hedging error by differentiating its price with respect to the strike price. The distribution from Monte Carlo simulation is compared with that obtained by our method.

AMS subject classifications: 91G20,91G60

Key words: Delta-hedging errors, profit and loss distribution, discrete trading, jump-diffusion model, transaction cost.

1. Introduction

In 1973 Black, Scholes & Merton [3, 10, 11] derived their famous equation for the price of a financial derivative. By calculating the price of replicating a portfolio under several assumptions — including continuous trading, no transaction cost, perfect dynamic hedging and stock price movement with geometric Brownian motion — an option trader can eliminate the randomness of the option price. Thus a trader can continuously replicate the option by adjusting a portfolio with an appropriate amount of stock and risk-free assets to remove its Delta-risk (the randomness caused by changes in the stock price), and this strategy is called 'Delta-hedging'.

However, historical data and many empirical results show that the Black–Scholes framework is imperfect and needs adjustment. For example, in practice there are transaction costs, we are not allowed to trade continuously, model parameters are unknown and seem to vary, and stock price movements are discontinuous. Thus perfect replication of a financial derivative is impossible, and there are errors caused by incomplete replication. Delta-hedging error, and the profit and loss (P&L) from imperfect Delta-hedging, has been studied

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extensively by many researchers under various conditions — e.g. unknown model parameters or discontinuous trading. Several authors [2,7,8] showed that the discrete time Delta-hedging error is proportional to $1/\sqrt{N}$, where N is the hedging frequency with equally distributed time intervals, and proposed a weak convergence of the Delta-hedging error. A more general convergence of the Delta-hedging error was considered by Tankov [15], and Bakshi & Kapadia [1] claimed that under Heston's stochastic volatility model the Delta-hedging error is driven by the option Vega and volatility risk premium. Cont *et al.* [6] and Mina [12] claim it is possible to hedge the Delta-hedging error with risk-free assets and additional options with the same underlying asset. Recently, Sepp [14] derived an approximated distribution of the Delta-hedging error using a characteristic function, which he found is approximated by the chi-squared distribution.

Option replicating, pricing and hedging with a transaction cost have been studied [4,9,13,16]. Boyle &Vorst [4] discussed option replication in a discrete-time framework with a transaction cost under a binomial option pricing model. Leland [9] claims that the volatility should change when the option price and its Greek parameters are calculated, if there is a transaction cost in the Black–Scholes framework. He also suggests a modified replicating strategy and Delta-hedging error under a discrete time framework with a transaction cost, and Toft [16] extended this to a derivation of a closed form for the expected Delta-hedging error.

We propose a new method to estimate the distribution of the Delta-hedging error with discrete time trading and transaction cost. In Section 2, we summarise the Black–Scholes framework with geometric Brownian motion and Merton's jump diffusion model, and state some basic facts about Delta-hedging strategy. In Section 3, we derive a recursive relation for option pricing with underlying P&L Delta-hedging error, and obtain its distribution by differentiating the option price twice with respect to the strike price. Numerical simulations presented in Section 4 are compared with distributions obtained by the Monte Carlo method, and our conclusions follow in Section 5.

2. Preliminaries

2.1. Black–Scholes framework

We assume that there exists a risk-neutral measure $\mathbb{Q}$ such that the discounted asset process $e^{-rt}S_t$ is $\mathbb{Q}$-martingale, where r is a risk-free rate. We say that the stock price process follows geometric Brownian motion (GBM) if S_t satisfies the stochastic differential equation (SDE)

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t \ , \tag{2.1}$$

where μ denotes the drift rate and σ is the volatility. From Ito's lemma, Eq. (2.1) may be rewritten

$$\mathrm{d}(\log S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}W_t \ ,$$