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Efficient and Stable Numerical Methods for Multi-Term Time Fractional Sub-Diffusion Equations

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Abstract. Some efficient numerical schemes are proposed for solving one-dimensional (1D) and two-dimensional (2D) multi-term time fractional sub-diffusion equations, combining the compact difference approach for the spatial discretisation and $L1$ approximation for the multi-term time Caputo fractional derivatives. The stability and convergence of these difference schemes are theoretically established. Several numerical examples are implemented, testifying to their efficiency and confirming their convergence order.

AMS subject classifications: 65M06, 65M12, 65M15

Key words: Multi-term time fractional sub-diffusion equations, compact /compact ADI difference scheme, discrete energy method, convergence.

1. Introduction

Fractional differential equations are now frequently invoked in various scientific and engineering applications. Physical phenomena in fields such as viscoelasticity, diffusion processes, relaxation vibrations and electrochemistry are successfully described by differential equations involving derivatives of fractional order [1–8]. Moreover, some underlying processes that cannot be described by single term time fractional partial differential equations can be described by multi-term equations — e.g. the multi-term time fractional diffusion-wave equation modelling various types of viscoelastic damping [9].

In this article, we provide some numerical difference schemes to solve multi-term time fractional sub-diffusion equations of the following form [9–11]:

$$P({}^{C}D_t)u(x,t) = \kappa \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)\,, \qquad 0 < x < L\,,\ 0 < t \le T\,, \tag{1.1}$$

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where κ is a positive diffusion constant and the multi-term fractional operator $P({}^C D_t)$ here is defined by

$$P({}^C D_t)v(x,t) = \left({}_0^C D_t^\alpha + \sum_{i=1}^{s} a_i \, {}_0^C D_t^{\alpha_i} \right) v(x,t)$$

with $0 < \alpha_s < \cdots < \alpha_1 < \alpha < 1$ and $a_i \in R$, $i = 1, 2, \cdots, s$, and ${}_0^C D_t^\alpha$ denotes the Caputo fractional derivative of order α:

$${}_0^C D_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v'(s)}{(t-s)^\alpha} ds \, .$$

We note that Luchko [10] obtained *a priori* solution estimates for the generalised multi-term time fractional diffusion equation using an appropriate maximum principle, and established solution uniqueness by the Fourier method. Daftardar-Gejji & Bhalekar [11] used separation of variables to solve a multi-term fractional diffusion-wave equation with homogeneous/nonhomogeneous boundary conditions. From Luchko's Theorem and the equivalent relationship between the Laplacian operator and the Riesz fractional derivative, Jiang *et al.* [12, 13] derived analytical solutions for both multi-term time-space fractional advection-diffusion and multi-term modified power law wave equations, respectively.

Other authors have discussed the numerical solution of fractional partial differential equations, including the fractional anomalous diffusion equation [14–24]. Chen *et al.* [14] proved the stability and convergence of an implicit difference approximation scheme of the fractional sub-diffusion equation using Fourier analysis. Lynch *et al.* [15] studied the numerical properties of the partial differential equations of fractional order $1 < \alpha < 2$. Yuste [16, 17] presented an explicit scheme and weighted average finite difference methods for the time fractional diffusion equation, and analysed their stability by the von-Neumann method. Zhuang *et al.* [18] obtained an implicit numerical method to solve the sub-diffusion equation by integrating the original equation on both sides, and proved the stability and convergence of their scheme by the energy method. Zhang *et al.* [19] constructed a Crank-Nicolson-type difference scheme and a compact difference scheme, to solve the time fractional sub-diffusion equation with a Riemann-Liouville fractional derivative. They proved that these two difference schemes are unconditionally stable, and the numerical solution converges in the maximum norm. Zhao & Sun [20] provided a box-type scheme for solving a class of fractional sub-diffusion equation with Neumann boundary conditions. Later, Ren *et al.* [21] proposed a compact difference scheme for the time fractional sub-diffusion equation with Neumann boundary conditions, and proved its unconditional stability and global convergence to be $O(\tau^{2-\alpha} + h^4)$ in the discrete L_2 norm.

There has also been some previous work on the numerical solution of problems with multiple fractional derivatives. Ford *et al.* [25] introduced a numerical method for solving the space-time fractional telegraph equation. Based on a quadrature formula approximation of the Caputo fractional derivative in spatial and temporal direction respectively, they proved the scheme was conditionally stable using the Fourier method. Liu *et al.* [26] proposed an implicit difference scheme for modified anomalous sub-diffusion equations with a nonlinear source term, and showed its convergence is $O(\tau + h^2)$ by the energy method.