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A Revisit of the Semi-Adaptive Method for Singular Degenerate Reaction-Diffusion Equations

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Abstract. This article discusses key characteristics of a semi-adaptive finite difference method for solving singular degenerate reaction-diffusion equations. Numerical stability, monotonicity, and convergence are investigated. Numerical experiments illustrate the discussion. The study reconfirms and improves several of our earlier results.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ denote a simply connected finite convex domain, consider a constant $b > 0$, and let u be sufficiently smooth in $\bar{\Omega}$. Many important multiphysics procedures, such as the n-dimensional quenching-combustion process, can be modelled ideally through the following singular reaction-diffusion initial-boundary value problem, or quenching problem:

$$\sigma(x)u_t = \nabla^2 u + f(u), \quad x \in \Omega, \quad t > t_0, \tag{1.1}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, \ t > t_0, \tag{1.2}$$

$$u(x,t_0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where ∇^2 is the Laplacian, $\partial\Omega$ is the boundary of Ω, $0 \le u_0 \ll b$, and

$$f(0) = f_0 > 0, \qquad f_u(u) > 0, \qquad u \in [0,b), \qquad \lim_{u \to b^-} f(u) = \infty.$$

The degeneracy function $\sigma(x) = 0$ for $x \in \Omega_0 \subset \partial\Omega$ [4, 6, 10, 11].

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It has been observed that when the shape of Ω is fixed there exists a critical number $a^* > 0$ such that if a (the n-volume of Ω) is less than a^* then the solution of Eqs. (1.1)–(1.3) exists globally. Otherwise, there exists a finite time $T^*(a)$ such that

$$\lim_{t\to T^*(a)} \sup_{x\in\Omega} u(x,t) = b.$$

Such an a^* is called a *critical value,* T^* a *critical time,* and b the *ignition temperature* [4, 10]. The function u is referred to as a *quenching solution* in the second case. The one-dimensional form of Eqs. (1.1)–(1.3) exhibits a particularly interesting example of the quenching phenomenon when $b = 1$, $\sigma \equiv 1$ and

$$f(u) = \frac{1}{1-u}, \tag{1.4}$$

where the critical value $a^* \approx 1.53045607591062$ [4, 18]. Recent investigations have also revealed that if solutions of Eqs. (1.1)–(1.3) exist they must increase monotonically as t increases at any fixed location $x \in \Omega$ [5, 10, 17].

We address the numerical solution of the one-dimensional form. Without any loss of generality we set $b = 1$ and map a general spatial interval $[s, s + a]$ to $[0, 1]$, and consequently consider the dimensionless problem

$$\sigma(x)u_t = \frac{1}{a^2}u_{xx} + f(u), \quad 0 < x < 1, \quad t_0 < t \le T, \tag{1.5}$$

$$u(0,t) = u(1,t) = 0, \qquad t > t_0, \tag{1.6}$$

$$u(x,t_0) = u_0(x), \qquad 0 < x < 1, \tag{1.7}$$

where $T < \infty$ is sufficiently large. The degeneracy and source functions of particular interest in multiphysics applications are

$$\sigma(x) = ax^p(1-x)^{1-p}, \quad f(u) = \frac{1}{(1-u)^q}, \quad 0 \le p \le 1, \quad q > 0, \tag{1.8}$$

for which the following limits are equivalent [5, 18]:

$$\lim_{t\to T_a} \sup_{0<x<1} u(x,t) = 1, \quad \lim_{t\to T_a} \sup_{0<x<1} u_t(x,t) = +\infty \quad \text{whenever } a > a^*.$$

This article is organised as follows. In Section 2 we introduce a second-order Crank-Nicolson scheme for solving Eqs. (1.5)-(1.8), where a uniform spatial mesh is used but adaptive grids controlled by a properly designed arc-length monitoring function are considered in the temporal direction. The structure and approximation is analysed, and both numerical stability in the von Neumann sense and nonlinear error propagation estimates are discussed. Motivated by the desire to preserve the most important physical characteristics of solutions, in Section 3 we focus on the monotonicity and convergence of the numerical solution sequence generated by the semi-adaptive finite difference scheme. Necessary constraints to ensure the correct multiphysical features are obtained, and we remark on the