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Two Parameter Iteration Methods for Coupled Sylvester Matrix Equations

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Abstract. We consider two parameter iteration methods for the coupled Sylvester matrix equations

$$
\begin{aligned}
AX + YB &= E\\
CX + YD &= F.
\end{aligned}
$$

Having established the convergence of the algorithms, we discuss the choice of optimal parameters. Numerical examples show the efficiency of the methods.

AMS subject classifications: 65F10, 15A24

Key words: Coupled Sylvester matrix equation, parameter iteration method, accelerating algorithm, convergence analysis.

1. Introduction

Let $\mathbb{R}^{m\times n}$ denote the set of $m\times n$ real matrices and let $\mathbb{R}^m := \mathbb{R}^{m\times 1}$. We consider the coupled Sylvester matrix equation

$$
\begin{aligned}
AX + YB &= E,\\
CX + YD &= F,
\end{aligned}
\tag{1.1}
$$

where $A, C \in \mathbb{R}^{m\times m}$, $B, D \in \mathbb{R}^{n\times n}$, $E, F \in \mathbb{R}^{m\times n}$ are constant matrices and $X, Y \in \mathbb{R}^{m\times n}$ are unknowns. Such equations find applications in various fields, including system theory [26, 46–48], control theory [39, 51], stability analysis [36], signal and image processing, photogrammetry. Thus in control systems and robust control [2, 13], one often encounters the continuous-time Sylvester equation $AX + XB = C$, discrete-time Sylvester equation $AXB^T + X = C$ and generalized Sylvester equation $AXB^T + CXD^T = F$. For the additive

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decomposition of a transfer matrix, Kågström and Dooren [24] proposed an algorithms based on the block-diagonalisation of $\lambda E - A$. Applying the QZ algorithm, they obtained a matrix $S_{A,Z}$ of the form

$$S_{A,Z}(\lambda) = \begin{bmatrix} \lambda E_{11} - A_{11} & \lambda E_{12} - A_{12} & B_1 \\ 0 & \lambda E_{22} - A_{22} & B_2 \\ -C_1 & -C_2 & D \end{bmatrix},$$

and the block $\lambda E_{12} - A_{12}$ can be eliminated by solving the generalized Sylvester equation

$$\begin{aligned} A_{11}R - LA_{22} &= -A_{12}, \\ E_{11}R - LE_{22} &= -E_{12}, \end{aligned}$$

for unknown matrices R and L. Another equation closely related to the matrix Sylvester equations is the second order system

$$A\ddot{x}(t) + B\dot{x}(t) + Cx(t) - Du(t) = 0,$$

which appears in vibration and structural analysis, spacecraft control and robotics [25].

The solvability of matrix equation has been widely studied — cf. Refs. [33–35, 52, 53] and the solution methods include the QR-factoring [13, 15, 31, 44], matrix splitting and gradient method [11, 12, 14, 40, 41], conjugate gradient method [1, 3–10, 17–20, 22, 23, 27, 37, 38, 42, 43, 45]. For the matrix equations of the form $AXB + C = E$, $AX + XB = F$, $AXB + CXD = F$, $A^TX + XA + B^TAX = C$, the parameter iteration methods [21, 50] can be used and it is the main tool we employ here. Therefore, in Section 2, we recall basic results needed for what follows. A parameter iteration method is described and investigated in Section 3. Section 4 deals with an accelerating algorithm for Eq. (1.1), and numerical examples are discussed in Section 5. Our final summary and conclusions are presented in Section 6.

2. Preliminaries

If A is a real matrix then A^T refers to the transpose of A. For any matrices $A = (a_{ij})$ and B we denote by $A \otimes B$ the Kronecker product of A and B — i.e. $A \otimes B := (a_{ij}B)$. If $X = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^{m\times n}$, then $\mathrm{vec}(X)$ denotes the mn column vector $(x_1^T, x_2^T, \cdots, x_n^T)^T$. Let us now recall a few basic results.

Lemma 2.1 (cf. Zhang [50]). *If $X \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{n\times n}$, then*

(i) $\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X)$.

(ii) If $\lambda(A)$ and $\lambda(B)$ are, respectively, the spectra of A and B, then

$$\lambda(A \otimes B) = \{\lambda_i \mu_j : \lambda_i \in \lambda(A), \quad \mu_j \in \lambda(B), \quad i = 1, 2, \cdots, m, \quad j = 1, 2, \cdots, n\}.$$