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The Unique Solvability and Approximation of BVP for a Nonlinear Fourth Order Kirchhoff Type Equation

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Abstract. We reduce a nonlinear fourth order equation of Kirchhoff type to an operator equation for a nonlinear term and establish sufficient conditions for the unique solvability of the original problem. Approximate solutions of the problem are derived by a fast converging iterative method. Numerical examples confirm theoretical results and demonstrate the efficiency of the approach used.

AMS subject classifications: 34B15, 34L30, 65L10, 65L12

Key words: Beam equation, Kirchhoff type equation, nonlinear equation, unique solvability, iterative method.

1. Introduction

We consider the following boundary value problem (BVP):

$$\begin{aligned} u^{(4)}(x) - M\left(\int_0^L |u'(s)|^2 ds\right) u''(x) \\ = f(x, u(x), u'(x), u''(x), u'''(x)), \quad 0 < x < L, \\ u(0) = u(L) = 0, \quad u''(0) = u''(L) = 0, \end{aligned} \tag{1.1}$$

which models the equilibrium of elastic beams. The term u in this equation denotes the deflection of an elastic beam of the length L and $f : [0, L] \times \mathbb{R}^4 \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous real valued functions. The integro-differential equation (1.1) is called the beam equation of the Kirchhoff type — cf. Ref. [10].

Numerous boundary value problems for the nonlinear equation

$$u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x))$$

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have been studied by different methods, including Leray-Schauder degree theory [11], Schauder fixed point theorem with lower and upper solutions [2, 6, 7, 9], Fourier analysis [8], and Banach fixed point theorem [5]. In addition, the boundary value problems for integro-differential equations of Kirchhoff type

$$u^{(4)}(x)-M\left(\int_0^L |u'(s)|^2 ds\right)u''(x)=f(x,u(x)), \qquad 0<x<L,$$
$$u^{(4)}(x)-M\left(\int_0^L |u'(s)|^2 ds\right)u''(x)=f(x,u(x),u'(x)),\ 0<x<L,$$

have been treated by variational and fixed point theory methods [12–15]. The unique solvability and iterative methods for Eq. (1.1) in the cases $L=\pi$, $M(x)=(2x)/\pi, f=p(x)+\epsilon u''(x)$ and $M(x)=0, f=f(x,u(x),u''(x))$ are, respectively, investigated in Refs. [3] and [4]. Amster and Cárdenas Alzate [1] studied the problem

$$\begin{aligned}
&u^{(4)}(x)-Au''(x)=f(x,u(x)), \quad 0<x<L,\\
&u(0)=a, \qquad u(L)=b,\\
&u''(0)=\alpha, \quad u''(L)=\beta,
\end{aligned}$$

by Leray-Schauder theorem under the assumption

$$\frac{f(x,u)-f(x,v)}{u-v}\le \overline{K}<\left(\frac{\pi}{L}\right)^2\left(\left(\frac{\pi}{L}\right)^2+A\right), \tag{1.2}$$

where $\overline{K}>0$ and A are constants. They also considered the equation

$$\begin{aligned}
&u^{(4)}(x)-A\int_0^L |u'(s)|^2 ds\, u''(x)=f(x,u(x)), \quad 0<x<L,\\
&u(0)=a,\ u(L)=b, \quad u''(0)=0,\ u''(L)=0,
\end{aligned}$$

under the assumptions that A is a constant and

$$f(x,u)=\overline{f}(x,u)+f_0(x,u), \tag{1.3}$$

with $\overline{f}$ and f_0 satisfying the conditions

$$\begin{aligned}
&\frac{\overline{f}(x,u)-\overline{f}(x,v)}{u-v}\le \overline{K}<\left(\frac{\pi}{L}\right)^4,\\
&|f_0(x,u)|\le \overline{k}|u|+l, \quad \overline{k}<\left(\frac{\pi}{L}\right)^4-\overline{K}, \quad \overline{k},l\in\mathbb{R}.
\end{aligned}$$

Recently, a novel method for fourth order nonlinear boundary problems based on their reduction to an operator equation for the right-hand side function f has been developed — cf. Refs. [4, 5]. In Section 2 we apply this method to the problem (1.1) and establish its unique solvability. Section 3 deals with an iterative method for the problem (1.1) and discusses a few numerical examples. It is worth mentioning that the right-hand side f used in these examples may not satisfy the requirements of [1].