A Chebyshev Spectral Collocation Method for Nonlinear Volterra Integral Equations with Vanishing Delays

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Abstract. A multistep Chebyshev-Gauss-Lobatto spectral collocation method for nonlinear Volterra integral equations with vanishing delays is developed. The convergence of the $hp$-version of the method in supremum norm is proved. Numerical experiments show the efficiency of the method for equations with highly oscillating, steep gradient and non-smooth solutions.

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1. Introduction

Spectral methods have a number of advantages and are widely used in approximate solution of partial differential equations — cf. Refs. [5, 6, 13, 14, 17, 19, 25, 26]. Recently, spectral methods have been employed in initial value problems for ordinary differential equations (ODEs) and delay differential equations (DDEs). For example, Guo and Wang, along with other researchers, [20, 21, 31, 33, 37–40] developed Legendre spectral collocation methods for ODEs and DDEs, and Wang et al. [23, 35] considered Chebyshev-Gauss and Chebyshev-Gauss-Lobatto spectral collocation methods for ODEs. In this work we investigate the Chebyshev-Gauss-Lobatto spectral collocation method for the following nonlinear

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Volterra integral equations (VIEs) with variable vanishing delays:

\[ y(t) = f(t) + \int_{0}^{t} K_1(t,s)G_1(s,y(s))ds + \int_{0}^{\theta(t)} K_2(t,s)G_2(s,y(s))ds, \]

where \( \theta(t) := t - \tau(t) \) is a delay function [9] such that

1. \( \tau(0) = 0, \tau(t) > 0 \) for \( t \in (0, T] \) (vanishing delay);
2. \( \theta(t) \in C^1(I) \) and \( \theta'(t) \geq q_0 > 0 \).

It is assumed that the kernels \( K_i \), \( i = 1, 2 \) belong to the spaces \( C(D_i) \) of continuous functions in the domains \( D_1 := \{(t,s) : 0 \leq s \leq t \leq T\} \), \( D_2 := \{(t,s) : 0 \leq s \leq \theta(t) \leq \theta(T)\} \), and the functions \( G_i \) are continuous on the corresponding subsets of \( \mathbb{R}^2 \) and \( f \in C(I) \). Note that the Eq. (1.1) contains VIEs with proportional delays \( \theta(t) = qt \), \( 0 < q < 1 \) studied earlier in Refs. [7, 15, 24].

Numerical methods for Volterra integral and differential equations with delays have been also considered. For example, piecewise polynomial collocation for equations with vanishing delays are discussed in [4, 7, 9–12, 34, 36], while piecewise polynomial collocation methods and Runge-Kutta methods for equations with non-vanishing delay are studied in [7–9, 30]. On the other hand, since spectral methods are very efficient in solving various problems with history, they become a popular tool in VIEs with delays — cf. [1–3, 18, 41]. However, the existing algorithms mainly comprise one-step methods, whereas for long time simulations, the multistep methods are more suitable. Several multistep methods have been considered recently: Conte and Paternoster [16] introduced a class of multistep collocation methods, each step of which uses Lagrange interpolation; Li et al. [22] proposed a time parareal method with spectral-subdomain enhancement for VIEs; Sheng et al. [27] considered a multistep spectral collocation method for nonlinear VIEs and established the convergence of the \( hp \)-version; Wang et al. [28, 32] developed a multistep Legendre-Gauss spectral collocation methods for nonlinear Volterra integral and Volterra functional integro-differential equations with delays.

In this work we develop an efficient multistep Chebyshev-Gauss-Lobatto spectral collocation method for Eq. (1.1). The main features of this work are:

- A multistep spectral collocation method for VIEs with nonlinear vanishing delays is considered. The existing works deal mainly with one-step methods with proportional or non-vanishing delays.
- Chebyshev expansions are used on each sub-step. The nodes and weights of Chebyshev-Gauss-Lobatto quadratures are given explicitly, which prevents a potential loss of accuracy. The algorithm can be implemented by exploiting the fast Chebyshev transform.
- The \( hp \)-convergence of the Chebyshev collocation method is fully analysed. A proper choice of \( h \) and \( p \) can significantly enhance the accuracy.