East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.210917.051217a

Vol. **8**, No. 2, pp. 224-232
May 2018

Abundant Mixed Lump-Soliton Solutions to the BKP Equation

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Received 21 September 2017; Accepted (in revised version) 5 December 2017.

Abstract. Applying Maple symbolic computations, we derive eight sets of mixed lump-soliton solutions to the $(2+1)$-dimensional BKP equation. The solutions are analytic and allow the separation of lumps and line solitons.

AMS subject classifications: 35Q51, 35Q53, 37K40

Key words: Lump solution, soliton solution, symbolic computation.

1. Introduction

It is well known that solitons describe various significant nonlinear phenomena in nature [1, 39] and the Hirota bilinear method provides a power tool for solving integrable equations [16]. Positons and complexitons are other typical solutions of integrable equations [25, 43], and the interaction between different classes of solutions leads to a better understanding of nonlinear phenomena [35]. In particular, the long wave limits of solitons generate lump solutions, rationally localized solutions in all directions in space, and many

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other ones — cf. Refs. [1,40]. Hirota bilinear forms play a crucial role in finding such exact solutions but the algorithm heavily relies on try and error experiments [5,16].

Let us recall [30] that the KP equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0,$$

has the following set of lump solutions:

$$\begin{aligned} u = 2(\ln f)_{xx},\ f = &\Big(a_1 x + a_2 y + \frac{a_1 {a_2}^2 - a_1 {a_6}^2 + 2a_2 a_5 a_6}{{a_1}^2 + {a_5}^2} t + a_4\Big)^2 \\ &+ \Big(a_5 x + a_6 y + \frac{2a_1 a_2 a_6 - {a_2}^2 a_5 + a_5 {a_6}^2}{{a_1}^2 + {a_5}^2} t + a_8\Big)^2 + \frac{3({a_1}^2 + {a_5}^2)^3}{(a_1 a_6 - a_2 a_5)^2}, \end{aligned}$$

where a_i are arbitrary parameters such that $a_1 a_6 - a_2 a_5 \neq 0$. This set contains a subset of lump solutions of the form

$$u = 4\frac{-[x + ay + (a^2 - b^2)t]^2 + b^2(y + 2at)^2 + 3/b^2}{\{[x + ay + (a^2 - b^2)t]^2 + b^2(y + 2at)^2 + 3/b^2\}^2}, \tag{1.1}$$

with two free parameters a and b [38]. The situation is not unique and there are many integrable equations with lump solutions — e.g. three-dimensional three-wave resonant interactions [21], the BKP equation [14, 45], the Davey-Stewartson equation II [40], the Ishimori-I equation [17] and so on. Besides, non-integrable equations, such as $(2+1)$-dimensional generalized KP, BKP and Sawada-Kotera equations, also have lump solutions — cf. Refs. [8,32,36,48,53].

It is worth noting that the general rational solutions of integrable equations have been derived within the framework of Wronskian, Casoratian, Grammian and Pfaffian formulations [1, 16]. The set of equations studied contains a variety of physically significant equations such as the KdV and Boussinesq equations, the nonlinear Schrödinger equation in $(1+1)$-dimensions, the KP and BKP equations in $(2+1)$-dimensions, and the Toda lattice equation in $(0+1)$-dimensions [2, 7, 13, 34, 35]. General rational solutions of nonlinear partial differential equations — e.g. generalized bilinear differential equations have been also discussed [3,37,47,50–52].

Here we consider a $(2+1)$-dimensional BKP equation of [9,18] — viz.

$$P_{BKP}(u) := (u_t + 15uu_{xxx} + 15{u_x}^3 - 15u_x u_y + u_{5x})_x - 5u_{xxxy} - 5u_{yy} = 0. \tag{1.2}$$

This is a first member in the BKP integrable hierarchy [6,41], represented by the $(2+1)$-dimensional generalization of the Caudrey-Dodd-Gibbon-Sawada-Kotera equation

$$v_t + 15vv_{xxx} + 15v_x v_{xx} + 45v^2 v_x + v_{5x} = 0. \tag{1.3}$$

If $v = u_x$ and the function u depends on x and t only, then (1.2) becomes the Eq. (1.3). The underlying spectral problem

$$-\phi_y + \phi_{xxx} + (3v - \lambda)\phi = 0, \tag{1.4}$$

has order 3. Hence one can apply the inverse scattering transform [1] to the Cauchy problem for the Eq. (1.3) — cf. Refs. [4, 20].

In this paper we present eight sets of interaction solutions of the $(2+1)$-dimensional BKP equation, obtained with the Maple symbolic computation software. These mixed lump-soliton solutions are an addition to the set of the lump and line soliton solutions known. Starting with the Hirota bilinear form of the $(2+1)$-dimensional BKP equation, we establish combinations of the hyperbolic cosine and quadratic functions satisfying the bilinear BKP equation.

2. Abundant Interaction Solutions

Let us start with the first-order logarithmic derivative transformation

$$u = 2(\ln f)_x, \tag{2.1}$$

routinely used in the Bell polynomial theory of integrable equations [15, 29]. Applying this transformation to the $(2+1)$-dimensional BKP equation (1.2), we transform it into the $(2+1)$-dimensional bilinear Hirota equation

$$\begin{aligned} B_{BKP}(f) :&= (D_x^6 - 5D_x^3 D_y + D_x D_t - 5D_y^2) f \cdot f \\ &= -10 f_{yy} f + 10 {f_y}^2 + 2 f_{xt} f - 2 f_t f_x - 10 f_{xxxy} f + 30 f_{xxy} f_x - 30 f_{xy} f_{xx} \\ &\quad + 10 f_y f_{xxx} + 2 f_{6x} f - 12 f_{5x} f_x + 30 f_{4x} f_{xx} - 20 {f_{xxx}}^2 = 0. \end{aligned} \tag{2.2}$$

Note that the $(2+1)$-dimensional BKP and bilinear BKP equations satisfy the relation

$$P_{BKP}(u) = \Big[\frac{B_{BKP}(f)}{f^2}\Big]_x. \tag{2.3}$$

Thus, if the function f is a solution of the bilinear BKP equation (2.2), then $u = 2(\ln f)_x$ is a solution of the BKP equation (1.2).

Our goal now is to find interaction solutions of the BKP equation (1.2) located between the lumps and line solitons and represented as a combination of the hyperbolic cosine and quadratic functions. In addition to the popular Hirota perturbation technique and symmetry constraints — cf. Refs. [10, 11, 22–24, 55], this approach amends basic tools for dealing with soliton and dromion-type solutions. More precisely, to discover such solutions, we employ the Maple computer algebra system starting with the combination

$$f = {\xi_1}^2 + {\xi_2}^2 + \cosh \xi_3 + a_{13}, \tag{2.4}$$

where the wave variables are defined by

$$\begin{aligned} \xi_1 &= a_1 x + a_2 y + a_3 t + a_4, \\ \xi_2 &= a_5 x + a_6 y + a_7 t + a_8, \\ \xi_3 &= a_9 x + a_{10} y + a_{11} t + a_{12}. \end{aligned} \tag{2.5}$$

The ansatz contains unknown real-valued parameters a_i, $i = 1, 2, \cdots, 13$.

In this work, the three special cases — viz. $a_9 = 0$, $a_{11} = 0$ and $a_{10} = 0$ are considered. Symbolic computations show that conditions $a_9 = 0$ and $a_{11} = 0$ do not produce any non-trivial solutions, whereas the case $a_{10} = 0$ generates eight sets of nontrivial solutions of the resulting algebraic systems. Other non-trivial solutions can be found in Ref. [33,46,49,54]. We note that if a parameter a_i is not specifically defined, then it can take any value within the corresponding solution set, provided that all expressions there make sense.

1. The set of solutions by solving in terms of the parameters a_1 and a_5:

$$a_2 = -\frac{1}{2}\frac{a_9{}^2(b\,a_5 - a_1{}^2 - a_5{}^2)}{a_1},\ a_3 = -\frac{5}{2}\frac{a_9{}^4(b\,a_5 + a_1{}^2 - a_5{}^2)}{a_1},\ a_6 = \frac{1}{2}\,b\,a_9{}^2,$$
$$a_7 = \frac{5}{2}a_9{}^4(b - 2\,a_5),\ a_{10} = 0,\ a_{11} = -a_9{}^5,\ a_{13} = -\frac{2\,a_1{}^4 + 4\,a_1{}^2a_5{}^2 + 2\,a_5{}^4 - a_9{}^4}{a_9{}^2(a_1{}^2 + a_5{}^2)},$$

where b is a solution of the equation $b^2 - 2\,a_5 b - 3\,a_1{}^2 + a_5{}^2 = 0$.

2. The set A of solutions by solving in terms of the parameters a_2 and a_6:

$$a_1 = -\frac{1}{2}\frac{b\,a_6 - a_2{}^2 - a_6{}^2}{a_2 a_9{}^2},\ a_3 = \frac{5}{2}\frac{a_9{}^2(b\,a_6 + a_2{}^2 - a_6{}^2)}{a_2},\ a_5 = \frac{1}{2}\frac{b}{a_9{}^2},$$
$$a_7 = -\frac{5}{2}\,b\,a_9{}^2 + 5\,a_6 a_9{}^2,\ a_{10} = 0,\ a_{11} = -a_9{}^5,\ a_{13} = -\frac{2\,a_2{}^4 + 4\,a_2{}^2a_6{}^2 + 2\,a_6{}^4 - a_9{}^{12}}{a_9{}^6(a_2{}^2 + a_6{}^2)},$$

where b is a solution of the equation $b^2 - 2\,a_6 b - 3\,a_2{}^2 + a_6{}^2 = 0$.

3. The set B of solutions by solving in terms of the parameters a_2 and a_6:

$$a_1 = \frac{1}{2}\frac{b}{a_9{}^2},\ a_3 = -\frac{5}{2}(b - 2\,a_2)a_9{}^2,\ a_5 = \frac{1}{2}\frac{a_6(b - 4\,a_2)}{a_9{}^2(b - a_2)},\ a_7 = \frac{5}{2}\frac{a_6 a_9{}^2(b + 2\,a_2)}{b - a_2},$$
$$a_{10} = 0,\ a_{11} = -a_9{}^5,\ a_{13} = -\frac{2\,a_2{}^4 + 4\,a_2{}^2a_6{}^2 + 2\,a_6{}^4 - a_9{}^{12}}{a_9{}^6(a_2{}^2 + a_6{}^2)},$$

where b is a solution of the equation $b^2 - 2\,a_2 b + a_2{}^2 - 3\,a_6{}^2 = 0$.

4. The set of solutions by solving in terms of the parameters a_3 and a_7:

$$a_1 = \frac{1}{10}\frac{b - 2\,a_3}{a_9{}^4},\ a_2 = \frac{1}{10}\frac{b}{a_9{}^2},\ a_5 = -\frac{1}{10}\frac{b\,a_3 - a_3{}^2 + a_7{}^2}{a_7 a_9{}^4},\ a_6 = \frac{1}{10}\frac{a_7(b - 4\,a_3)}{a_9{}^2(b - a_3)},$$
$$a_{10} = 0,\ a_{11} = -a_9{}^5,\ a_{13} = -\frac{1}{25}\frac{2\,a_3{}^4 + 4\,a_3{}^2a_7{}^2 + 2\,a_7{}^4 - 625\,a_9{}^{20}}{(a_3{}^2 + a_7{}^2)a_9{}^{10}},$$

where b is a solution of the equation $b^2 - 2\,a_3 b + a_3{}^2 - 3\,a_7{}^2 = 0$.

5. The set A of solutions by solving in terms of the parameters a_1 and a_6:

$$a_2 = a_1 b^2,\ a_3 = 0,\ a_5 = -\frac{a_6}{b^2},\ a_7 = 10\,a_6 b^2,\ a_9 = b,$$
$$a_{10} = 0,\ a_{11} = -\frac{3\,b\,a_6{}^2}{a_1{}^2},\ a_{13} = -\frac{1}{12}\frac{32\,a_1{}^6 - 27\,a_6{}^2}{a_1{}^4 b^2},$$

where b is a solution of the equation $a_1{}^4 b^4 - 3a_6{}^2 = 0$.

6. The set B of solutions by solving in terms of the parameters a_1 and a_6:

$$a_2 = \frac{2a_1{}^2 a_9{}^4 + b\,a_6 - 2a_6{}^2}{a_1 a_9{}^2},\ a_3 = \frac{5(a_1{}^2 a_9{}^4 + b\,a_6 - 2a_6{}^2)}{a_1},$$

$$a_5 = \frac{b}{a_9{}^2},\ a_7 = -5\,b\,a_9{}^2 + 5a_6 a_9{}^2,\ a_{10} = 0,\ a_{11} = -a_9{}^5,$$

$$a_{13} = -\frac{1}{4}\,\frac{32a_1{}^4 a_9{}^8 - a_9{}^{12} + 64\,b\,a_1{}^2 a_6 a_9{}^4 + 32a_1{}^2 a_6{}^2 a_9{}^4 + 64\,b\,a_6{}^3 - 96a_6{}^4}{a_9{}^6(a_1{}^2 a_9{}^4 + b\,a_6 - a_6{}^2)},$$

where b is a solution of the equation $b^2 - 4a_6 b + 4a_6{}^2 - 3a_1{}^2 a_9{}^4 = 0$.

7. The set A of solutions by solving in terms of the parameters a_1 and a_7:

$$a_2 = a_1 b^2,\ a_3 = 0,\ a_5 = -\frac{1}{10}\frac{a_7}{b^4},\ a_6 = \frac{1}{10}\frac{a_7}{b^2},\ a_9 = b,$$

$$a_{10} = 0,\ a_{11} = -b^5,\ a_{13} = -\frac{1}{12}\,\frac{32a_1{}^4 - 9b^4}{a_1{}^2 b^2},$$

where b is a solution of the equation $100a_1{}^2 b^8 - 3a_7{}^2 = 0$.

8. The set B of solutions by solving in terms of the parameters a_1 and a_7:

$$a_2 = \frac{1}{25}\,\frac{a_7 b + 2a_7{}^2 - 25a_1{}^2 a_9{}^8}{a_1 a_9{}^6},\ a_3 = \frac{1}{5}\,\frac{a_7 b + 2a_7{}^2 - 50a_1{}^2 a_9{}^8}{a_1 a_9{}^4},$$

$$a_5 = \frac{1}{5}\frac{b}{a_9{}^4},\ a_6 = \frac{1}{5}\frac{b + a_7}{a_9{}^2},\ a_{10} = 0,\ a_{11} = -a_9{}^5,$$

$$a_{13} = -\frac{1}{100}\,\frac{625a_9{}^{20} + 1600a_1{}^2 a_7 a_9{}^8 b - 800a_1{}^2 a_7{}^2 a_9{}^8 + 64a_7{}^3 b + 96a_7{}^4 - 20000a_1{}^4 a_9{}^{16}}{a_9{}^{10}(a_7 b + a_7{}^2 - 25a_1{}^2 a_9{}^8)},$$

where b is a solution of the equation $b^2 + 4a_7 b + 4a_7{}^2 - 75a_1{}^2 a_9{}^8 = 0$.

It is clear that all the quadratic equations above have real solutions, and so all sets of parameters a_j are well defined. These sets generate eight classes of the combination solutions (2.4)-(2.5) to the bilinear BKP equation (2.2). Subsequently, the transformation (2.1) produces eight classes of mixed lump-soliton solutions of the $(2+1)$-dimensional BKP equation (1.2) and for positive a_{13}, the interactions solutions are analytic.

The interaction solutions become, respectively, line or lump solutions depending on whether the quadratic function or the hyperbolic cosine disappears. Let us emphasize that the interaction solutions found contain a line soliton solution. Therefore, they do not approach zero in all directions in the space of both spatial and temporal variables. Moreover, due to the presence of a lump solution, they form a peak at a finite time.

The graph of the interaction solution of the $(2+1)$-dimensional BKP equation (1.2) for the parameters $a_{10} = 0$, $b = 1 + \sqrt{3}$ and

$$\begin{aligned} &a_1 = -1, && a_2 = 2\sqrt{3} - 2, && a_3 = 40 + 40\sqrt{3}, && a_4 = 1,\\ &a_5 = 1, && a_6 = 2 + 2\sqrt{3}, && a_7 = 40\sqrt{3} - 40, && a_8 = -1,\\ &a_9 = 2, && a_{11} = -32, && a_{12} = 2, && a_{13} = 1, \end{aligned} \tag{2.6}$$

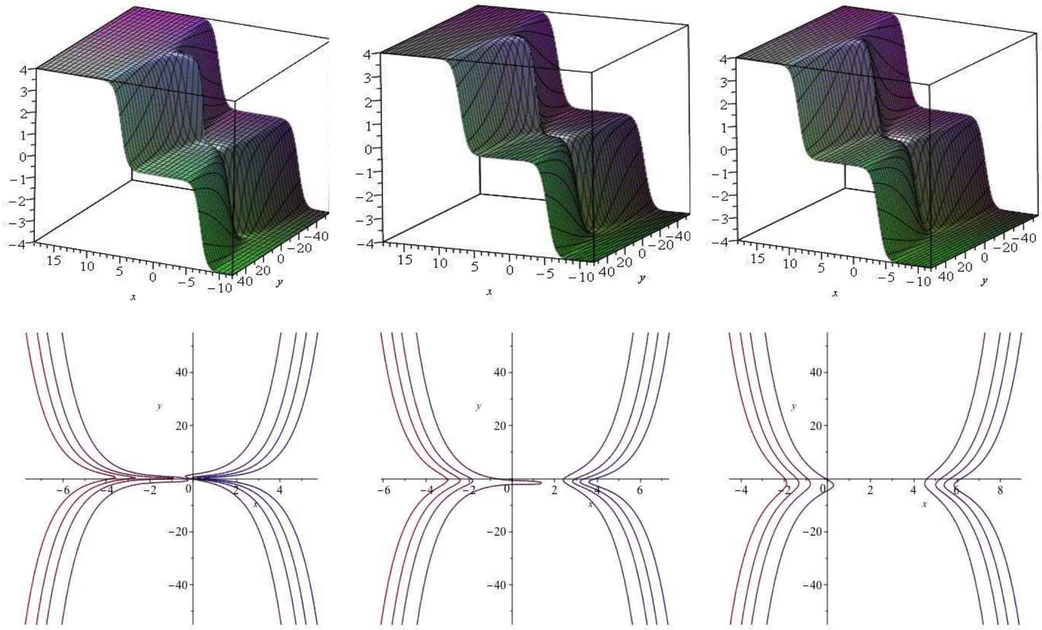

Figure 1: Solution of (2.6) for $t=0$, $t=0.1$ and $t=0.2$. Top: 3d plots. Bottom: Contour plots.

is presented in Fig. 1.

We also tried to solve the resulting systems of nonlinear algebraic equations with other combinations of parameters but have not found any new non-trivial solutions.

3. Concluding Remarks

Applying Maple symbolic computations, we found eight sets of exact mixed lump-soliton solutions of the $(2+1)$-dimensional BKP equation. These solutions generate lump and line soliton solutions as special cases, and the technique adopted above provides a supplement to the different combination theories in [31, 42, 44, 56]. It would be interesting to know which of these new solutions can be detected by the Wronskian technique [12, 19].

Let us also note that the systems of nonlinear algebraic equations connected with the BKP equation, are usually not solvable in general setting but only for specific sets of parameters. Combination solutions of generalized bilinear and tri-linear differential equations with generalized bilinear derivatives [26] are also of interest, since the corresponding mixed interaction solutions are different from the resonant ones generated via linear superposition principle [27, 28]. In particular, the BKP-like equation defined with $p=3$,

$$(D_{3,x}^6-5D_{3,x}^3D_{3,y}+D_{3,x}D_{3,t}-5D_{3,y}^2)f\cdot f=0,$$

has distinct lump-soliton solutions, whereas lump solutions derived from quadratic functions remain the same — cf. Ref. [37].

Acknowledgments

The work was supported in part by the Natural Science Foundation of Jiangsu Province (No. BK20151160), Shanghai Pujiang Program (No. 14PJD007), the Natural Science Foundation of Shanghai (No. 14ZR1403500), the Young Teachers Foundation (No. 1411018) of Fudan university, by NSFC (Grants Nos. 11371326, 11301331, 11571079 and 11371086), by NSF (Grant No. DMS-1664561), and by the Distinguished Professorships of Shanghai University of Electric Power and Shanghai Polytechnic University.

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