East Asian Journal on Applied Mathematics Vol. **3**, No. 2, pp. 107-119
doi: 10.4208/eajam.030513.200513a May 2013

Godunov Method for Stefan Problems with Enthalpy Formulations

D. Tarwidi[1,*] and S.R. Pudjaprasetya[2]

[1] *Department of Computational Science, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha 10, Bandung 40132, Indonesia.*
[2] *Industrial and Financial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha 10, Bandung 40132, Indonesia.*

Received 3 May 2013; Accepted (in revised version) 20 May 2013

Available online 31 May 2013

Abstract. A Stefan problem is a free boundary problem where a phase boundary moves as a function of time. In this article, we consider one-dimensional and two-dimensional enthalpy-formulated Stefan problems. The enthalpy formulation has the advantage that the governing equations stay the same, regardless of the material state (liquid or solid). Numerical solutions are obtained by implementing the Godunov method. Our simulation of the temperature distribution and interface position for the one-dimensional Stefan problem is validated against the exact solution, and the method is then applied to the two-dimensional Stefan problem with reference to cryosurgery, where extremely cold temperatures are applied to destroy cancer cells. The temperature distribution and interface position obtained provide important information to control the cryosurgery procedure.

AMS subject classifications: 65M10, 78A48

Key words: Stefan problems, Godunov method, solidification, enthalpy, cryosurgery.

1. Introduction

Stefan problems describe phase change moving boundaries, such as in solidification and melting processes. Their main characteristic is that the location of the interface between two phases is unknown, and must be determined as part of the solution. After Josef Stefan compared his calculations for the melting of the polar ice cap with the existing observational data around 1890, Stefan problems were soon found to be important in many other areas of the natural sciences and elsewhere. In industrial processes, Stefan problems

*Corresponding author. *Email addresses:* dede.tarwidi@yahoo.com (D. Tarwidi), sr_pudjap@math.itb.ac.id (S.R. Pudjaprasetya)

occur in metal solidification, food freezing, and ice production. In medical science, the Stefan problem arises in cryosurgery, where in particular cancer cells may be destroyed under extremely cold temperatures.

In Stefan problems where the heat conduction equation is to be solved in both solid and liquid regions, the moving boundary or interface that separates the two regions presents a major difficulty. Analytical solutions are very limited, even for one-dimensional problems. Alexiades & Solomon [1] have discussed in detail the analytical and numerical solution of one-dimensional and two-dimensional Stefan problems.

There have been many methods developed to solve moving boundary problems more generally, including the enthalpy method, the boundary immobilisation method and perturbation, nodal integral and heat balance integral methods [2]. However, the enthalpy method is the most widely used in solving Stefan problems [3–5], as its strength lies in reformulating the heat conduction equations to involve the internal energy (enthalpy). Thus in the enthalpy reformulation the governing equation stays the same for any phase — whether solid, liquid, or even gas. In corresponding discrete formulations, the conservative property of the system is directly preserved in the difference equations. In particular, the finite volume method can thereby simulate the discontinuous solutions with correct speeds, and automatically predict the moving interfaces.

In this article, the first-order Godunov method is adopted to solve the Stefan problem. Comprehensive reviews of the Godunov method can be found in Refs. [6–10]. In Section 2, the exact solution of the one-dimensional solidification problem is used to test our implementation, which is then applied to simulate the two-dimensional system in Section 3.

2. One-Dimensional Stefan Problem

In this section, we first discuss the mathematical formulation and the analytical solution of the one-dimensional Stefan problem, and then compare the results we obtain using the Godunov method in the enthalpy formulation.

2.1. Mathematical formulation and analytical solution

Consider a one-dimensional container of length l, full of liquid with a freezing temperature T_m. Suppose the initial temperature of the liquid T_L is higher than T_m, and one end of the liquid $x = 0$ is maintained at temperature $T_S(< T_m)$ for $t > 0$, whereas the other end $x = l$ is insulated. The solidification process consequently starts from $x = 0$, and extends over increasing intervals as the time t increases (a well-known Stefan problem). We assume that the material density ρ is constant; and the thermophysical properties are the latent heat L, the respective specific heats of the liquid and solid c_L and c_S, and the respective thermal conductivities of the liquid and solid k_L and k_S.

Suppose $X(t)$ is the interface that separates the two regions at time t, such that $0 \leq x < X(t)$ is the solid region and $X(t) < x \leq l$ is the liquid region — cf. Fig. 1. Our aim is to determine the temperature distribution $T(x,t)$ throughout the material, and the interface position $X(t)$. Heat conduction in the solid and liquid regions obeys the respective