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Preconditioned Positive-Definite and Skew-Hermitian Splitting Iteration Methods for Continuous Sylvester Equations $AX + XB = C$

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Abstract. In this paper, we present a preconditioned positive-definite and skew-Hermitian splitting (PPSS) iteration method for continuous Sylvester equations $AX + XB = C$ with positive definite/semi-definite matrices. The analysis shows that the PPSS iteration method will converge under certain assumptions. An inexact variant of the PPSS iteration method (IPPSS) has been presented and the analysis of its convergence property in detail has been discussed. Numerical results show that this new method is more efficient and robust than the existing ones.

AMS subject classifications: 65F10, 65F50

Key words: PPSS iteration method, IPPSS iteration method, Sylvester equations, convergence.

1. Introduction

In this paper, we consider the iteration solution of the following continuous Sylvester equation of the form

$$AX + XB = C, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{m\times n}$. Assume that
(A_1) A, B, C are large and sparse matrices;
(A_2) at least one of A and B is non-Hermitian;
(A_3) both A and B are positive semi-definite, and at least one of them is positive definite.

It is well known that the continuous Sylvester equation (1.1) has a unique solution if and only if there is no common eigenvalue between A and $-B$. Note that a Lyapunov equation is a special case of the continuous Sylvester equation with $B = A^*$ and C being

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Hermitian. Here and in the sequel, A^* represents the conjugate transpose of the matrix A. Both Lyapunov and Sylvester equations have numerous applications. We can obtain the history of this class of equations and many interesting and important theoretical results in [2]. The continuous Sylvester equation (1.1) has numerous applications in control and system theory [32], signal processing [1], model order reduction [31], image restoration [19], stability of linear systems [26], analysis of bilinear systems [29], power systems [27], linear algebra [23], numerical methods for differential equations [2, 9, 10, 12, 13, 16, 17], matrix nearness problem [30], finite element model updating [22], block-diagonalization of matrices [23] and so on. Many of these applications lead to stable Sylvester equations, i.e., Assumption (A_3) made in the above is satisfied. Therefore, how to effectively solve this kind of equations involving literally hundreds or thousands of variables is under research.

The continuous Sylvester equation (1.1) is mathematically equivalent to the system of linear equations

$$\mathscr{A} x = c, \tag{1.2}$$

where $\mathscr{A} = I \otimes A + B^T \otimes I$, and the vectors x and c contain the concatenated columns of the matrices X and C respectively, with $\otimes$ being the Kronecker product symbol. Of course, this is a numerically poor way to determine the solution X of the continuous Sylvester equation (1.1), as the system of linear equations (1.2) is costly to solve and could be ill-conditioned.

Standard direct methods for numerical solution of the continuous Sylvester equation (1.1) are the Bartels-Stewart and the Hessenberg-Schur methods [24], which consist in transforming A and B into triangular or Hessenberg-Schur form by an orthogonal similarity transformation and then solving the resulting system of linear equations directly by a back-substitution process. However, they are not applicable and too expensive for large-scale problems. For large-scale continuous Sylvester equations, iterative methods have been developed by taking advantage of the sparsity and the low-rank structure of C. The Alternating Direction Implicit (ADI) method [3, 35, 36] and the Krylov subspace based algorithms [6, 21, 25, 28, 33] are the most common iterative methods. Advantages of Krylov subspace based algorithms over ADI iterations are that no knowledge about the spectra of A and B is needed and (except for [33]) no linear systems of equations with (shifted) A and B have to be solved, and but ADI iterations often enable faster convergence if optimal shifts to A and B can be effectively estimated [4].

The HSS iteration method for system of linear equations was firstly proposed by Bai, Golub and Ng in [7], and then it was extended to other equations and conditions in [7–18, 34, 37–40]. Recently, Bai in [5] proposed a Hermitian and skew-Hermitian splitting (HSS) iteration method for solving large sparse continuous Sylvester equations with non-Hermitian and positive definite/semidefinite matrices. In [37], Wang et al. applied the idea of PSS iteration method in [6] to solve the continuous Sylvester equations and in [41]. Zheng and Ma applied the NSS iteration method [4] to solve the continuous Sylvester equations. Recently, Dong and Gu presented a PMHSS iteration method [20] for Sylvester equations. Motivated by this, we further present and analyze a preconditioned positive definite and skew-Hermitian iteration method for solving the continuous Sylvester equations, which is called PPSS iteration method.