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Optimal Production Control in Stochastic Manufacturing Systems with Degenerate Demand

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Abstract. The paper studies the production inventory problem of minimizing the expected discounted present value of production cost control in manufacturing systems with degenerate stochastic demand. We have developed the optimal inventory production control problem by deriving the dynamics of the inventory-demand ratio that evolves according to a stochastic neoclassical differential equation through Ito's Lemma. We have also established the Riccati based solution of the reduced (one- dimensional) HJB equation corresponding to production inventory control problem through the technique of dynamic programming principle. Finally, the optimal control is shown to exist from the optimality conditions in the HJB equation.

AMS subject classifications: 49N05, 60H10, 90B30, 90B05

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1. Introduction

Let us consider the dynamics of the state equation which says that the inventory at time t is increased by the production rate and decreased by the demand rate, and can be written as

$$dI(t) = [k(t) - D(t)]dt, \quad I(0) = I, \ \ I > 0, \tag{1.1}$$

and the demand equation with the production rate is described by the Brownian motion

$$dD(t) = AD(t)dt + \sigma D(t)dw(t), \quad D(0) = D, \ \ D > 0. \tag{1.2}$$

Here $I(t)$ is the inventory level for production rate at time t (state variable), $D(t)$ is the demand rate at time t, A is a non-zero constant, σ is the non-zero constant diffusion coefficient, $k(t) \geq 0$ represents the production rate at time t (control variable), w_t is

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a one-dimensional standard Brownian motion on a complete probability space $(\Omega, \mathscr{F}, P)$ endowed with the natural filtration $\mathscr{F}_t$ generated by $\sigma(w(s), s \leq t)$, $I(0)$ is the initial value of inventory level and $D(0)$ is the initial value of the demand rate.

In this paper, the optimization problem involves minimizing the expected discounted cost control

$$J(k(t)) = E\left[\int_0^\infty e^{-\beta t}\left\{f(I(t)) + k(t)^2\right\} dt\right] \tag{1.3}$$

over $k \in \mathscr{K}$ for all $t \geq 0$ where, $\mathscr{K}$ denotes the class of all admissible controls of production processes, $\beta > 0$ is the constant non-negative discount rate.

We assume f is a continuous, non-negative, convex function satisfying the polynomial growth condition such that

$$0 \leq f(I(t)) \leq L(1 + |I(t)|^n), \quad I(t) \in \mathbf{R}, \quad \forall t \geq 0, \quad n \in \mathbf{N}_+ \tag{1.4}$$

for some constant $L > 0$.

By the Principle of Optimality, it is natural that u solves the general (two-dimensional) Hamilton Jacobi-Bellman (HJB) equation

$$\begin{aligned} &-\beta u(I(t), D(t)) + \frac{1}{2}\sigma^2 D(t)^2 u_{DD} + AD(t)u_D - D(t)u_I + F^*(u_I) + h(I(t)) = 0, \\ &u(0, D) = 0, \quad I(t) > 0, D(t) > 0, \end{aligned} \tag{1.5}$$

where $F^*(u_I) = \min_{k(t)\geq 0}\{k(t)^2 + k(t)u_I\}$, u_I, u_D, u_{DD} are partial derivatives of $u(I(t), D(t))$ with respect to $I(t)$ and $D(t)$, and $F^*(I)$ is the Legendre transform of $F(I)$, - i.e. $F^*(I(t)) = \min_{k\geq 0}\{k^2(t) + k(t)I(t)\}$.

There exists a $v \in (0, \infty)$ such that $u(I(t), D(t)) = D(t)^2 v(I(t)/D(t)), D(t) > 0$. Since

$$\begin{aligned} u_I &= D(t)v'(I(t)/D(t)), \\ u_D &= 2D(t)v(I(t)/D(t)) - I(t)v'(I(t)/D(t)), \\ u_{DD} &= 2v(I(t)/D(t)) - 2(I(t)/D(t))v'(I(t)/D(t)) + (I(t)/D(t))^2 v''(I(t)/D(t)), \end{aligned}$$

on setting $z(t) = I(t)/D(t)$ and substituting into (1.5) yields, we have

$$\begin{aligned} &-\tilde{\beta} v(z(t)) + \frac{1}{2}\sigma^2 z^2(t) v''(z(t)) + \tilde{A} z(t) v'(z(t)) + F^*(v'(z(t))) + f(z(t)) = 0, \\ &v(0) = 0, \quad z(t) > 0, \forall t \geq 0 \end{aligned} \tag{1.6}$$

where $\tilde{\beta} = -\beta + \sigma^2 + 2A$, $\tilde{A} = -A + \sigma^2$, $F^*(v'(z(t))) = \min_{p(t)\geq 0}\{(p(t)+1)^2 + p(t)v'(z(t))\}$, and $F^*(z(t))$ is the Legendre transform of $F(z(t))$ - i.e. $F^*(z(t)) = \min_{p(t)\geq 0}\{(p(t)+1)^2 + p(t)z(t)\}$.

The control problem of production planning in manufacturing systems with discount rate has been studied by many authors - e.g. Fleming, Sethi and Soner (1987), Sethi and Zhang (1994). The Bellman equation associated with the production inventory control