Commun. Comput. Phys.
doi: 10.4208/cicp.090210.040610a

Vol. 9, No. 4, pp. 1071-1080
April 2011

SHORT NOTE

On the Connection Between the Spectral Difference Method and the Discontinuous Galerkin Method

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Received 9 February 2010; Accepted (in revised version) 4 June 2010

Communicated by Kun Xu

Available online 10 November 2010

Abstract. In this short note we present a derivation of the Spectral Difference Scheme from a Discontinuous Galerkin (DG) discretization of a nonlinear conservation law. This allows interpretation of the Spectral Difference Scheme as a particular discretization under the quadrature-free nodal DG paradigm. Moreover, it enables identification of the key differences between the Spectral Difference Scheme and standard nodal DG schemes.

AMS subject classifications: 65N30, 65N99, 65M60, 65M99

Key words: Spectral difference method, discontinuous Galerkin method, hyperbolic conservation laws, quadrature-free schemes, nodal schemes.

1 Introduction

High-order numerical schemes that are based on locally discontinuous polynomial approximations on standard unstructured meshes are particularly attractive for nonlinear convection-dominated problems in complex geometry. The Discontinuous Galerkin (DG) method [1] is a well-known example. If accuracy requirements are moderate, however, higher order schemes are advantageous only if a particular discretization method supports efficient implementation. Local numerical volume and surface integration required by the DG approach can be considered a drawback in this regard. For nonlinear conservation laws, where integration necessitates explicit evaluation of analytical and numerical flux functions at quadrature points, numerical quadratures are quite irksome, since a generic optimal node placement for a given accuracy is not known for general mesh

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elements, in particular simplex elements. One often uses integration rules based on (singular) tensor products, which oversample the solution in order to achieve the desired order of accuracy [2].

In this context the Spectral Difference Scheme [3–5] has been proposed as a collocation-based method, using local interpolation of the strong form of the equation, with the aim to achieve superior efficiency by avoiding volume and surface quadratures altogether, while maintaining conservation. The Spectral Difference approach extends tensor-product-based collocation approaches that had previously been formulated for quadrilateral meshes [6] to more general unstructured-grid elements.

Another approach is given by "quadrature-free" DG schemes [7] that avoid numerical integration by a suitable projection of the nonlinear analytical and numerical flux functions in the interior and on the surface of mesh elements, respectively, onto finite-dimensional spaces. As a consequence all integrations involve only analytically known basis functions, which allows exact evaluation. This requires fewer degrees of freedom compared to suboptimal numerical quadrature.

In this paper we demonstrate that in fact the Spectral Difference Scheme for nonlinear hyperbolic conservation laws and a particular nodal (quadrature-free) Discontinuous Galerkin scheme are equivalent to each other under certain well-defined conditions. More specifically, the Spectral Difference Scheme is obtained from previously documented nodal DG schemes [8–10] by using the numerical flux function in the quadrature-free discretization of the volume integrals, whereas in the traditional nodal DG approach only analytical flux function evaluations are used for that purpose. Furthermore, the Spectral Difference Scheme is identified as a particulary efficient scheme among the class of quadrature-free DG schemes. It is hoped that establishing the variational formulation of the Spectral Difference Scheme may be useful for further theoretical analysis, perhaps allowing previously established results for nodal DG schemes to be re-used.

Unifying treatment for discretizations using locally discontinuous polynomial approximations has recently been advanced in a more general context by Wang et al. in their formulation of lifting collocation penalty methods [11]. The quadrature-free paradigm for DG methods used here may be the necessary ingredient of extending such a unifying formulation in a clean way to nonlinear equations.

The paper is organized as follows: we briefly and rather informally recall the definition of the Spectral Difference Scheme in the classical derivation from the strong form of the governing equations in Section 2. Subsequently we demonstrate in Section 3 the essential steps for the derivation from the variational formulation for a nonlinear one-dimensional conservation law. It remains to incorporate the more complicated metric structure in the multi-dimensional case, which is considered in Section 4.

2 The Spectral Difference Scheme

Consider the scalar hyperbolic conservation law