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Numerical Soliton Solutions for a Discrete Sine-Gordon System

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Abstract. In this paper we use an analytical-numerical approach to find, in a systematic way, new 1-soliton solutions for a discrete sine-Gordon system in one spatial dimension. Since the spatial domain is unbounded, the numerical scheme employed to generate these soliton solutions is based on the artificial boundary method. A large selection of numerical examples provides much insight into the possible shapes of these new 1-solitons.

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1 Introduction

The sine-Gordon equation,

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+\sin(u)=0 \tag{1.1}$$

is a semilinear hyperbolic equation in $1+1$ dimensions. This PDE has its origin in the 19th century where it arose in the study of surfaces of constant negative curvature (cf. [1]). In the second half of the 20th century the sine-Gordon equation has attracted considerable attention, owing to its importance in the mathematical modeling of various physical phenomena, for example in nonlinear optics (propagation of pulses in resonant media); superconductivity (wave propagation in a Josephson transmission line); condensed matter

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physics (charge density waves in periodic pinning potentials); and in solid state physics (propagation of a dislocation in a crystal). Details and additional examples can be found in [2–5].

A very important property of the sine-Gordon equation (1.1) is the existence of soliton solutions. Many of these special solutions have been obtained in closed form, by using analytical methods such as Bäcklund transformations [6], the nonlinear separation of variables method [2]; see also [3] (Chapter 6). The known soliton solutions of (1.1) mainly can be classified as follows:

1. 1-soliton solutions : Two 1-soliton solutions are given by [7,8]

$$u(x,t)=4\arctan\left\{\exp\left(\pm\frac{x-\mu t-x_0}{\sqrt{1-\mu^2}}\right)\right\},\quad \mu^2<1, \tag{1.2}$$

and

$$u(x,t)=-\pi+4\arctan\left\{\exp\left(\pm\frac{x-\mu t-x_0}{\sqrt{\mu^2-1}}\right)\right\},\quad \mu^2>1. \tag{1.3}$$

Here, $x_0,\mu\in\mathbb{R}$ and $|\mu|\neq 1$.

2. Breather solutions: Two breather solutions to (1.1) are given by [7,8]

$$u(x,t)=4\arctan\left\{\frac{\mu\sinh(kx+A)}{k\cosh(\mu t+B)}\right\},\quad \mu^2=k+1, \tag{1.4}$$

and

$$u(x,t)=4\arctan\left\{\frac{\mu\sin(kx+A)}{k\cosh(\mu t+B)}\right\},\quad \mu^2=1-k^2>0. \tag{1.5}$$

Here, A and B are arbitrary (real) constants, and the real numbers μ and k are related by the conditions in (1.4) and (1.5), respectively.

3. N-soliton solutions: An N-soliton solution for (1.1) is given by

$$u(x,t)=x+\arccos\left\{1-2\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}\right)\ln F(x,t)\right\}, \tag{1.6}$$

with

$$\begin{aligned} &F(x,t):=\det[M_{ij}], \qquad M_{ij}:=\frac{2}{a_i+a_j}\cosh\left(\frac{z_i+z_j}{2}\right),\\ &z_i:=\pm\frac{x-\mu_i t+C_i}{\sqrt{1-\mu_i^2}}, \qquad a_i:=\pm\sqrt{\frac{1-\mu_i^2}{1+\mu_i^2}}. \end{aligned}$$